Regularity for a minimum problem with free boundary in Orlicz spaces

Jun Zheng [1], Leandro S. Tavares[2], Claudianor O. Alves[3]

[1] School of Mathematics, Southwest Jiaotong University, Chengdu 611756, Sichuan, China
[2] Centro de Ciências e Tecnologia, Universidade Federal do Cariri, Avenida Tenente Raimundo Rocha, CEP 58429-900, Juazeiro do Norte, CE, Brazil
[3] Unidade Acadêmica de Matemática, Universidade Federal de Campina Grande, CEP 58429-900, Campina Grande, PB, Brazil

Abstract

The aim of this paper is to study the heterogeneous optimization problem

$$\mathcal{J}(u)=\int_\Omega (G(|\nabla u|)+qF(u^+)+hu+\lambda_+\chi_{\{u>0\}})\mathrm{d}x \to \min,$$

in the class of functions $W^{1,G}(\Omega)$ with $u-\varphi \in W_0^{1,G}(\Omega)$, for a given function φ, where $W^{1,G}(\Omega)$ is the class of weakly differentiable functions with $\int_\Omega G(|\nabla u|)\mathrm{d}x<\infty$. The functions G and F satisfy structural conditions of Lieberman's type that allow for a different behavior at 0 and at ∞. Given functions q,h and constant $\lambda_+\geq 0$, we address several regularities for minimizers of $\mathcal{J}(u)$, including local $C^{1,\alpha}-$, and local Log-Lipschitz continuities for minimizers of $\mathcal{J}(u)$ with $\lambda_+=0$, and $\lambda_+>0$ respectively. We also establish growth rate near the free boundary for each non-negative minimizer of $\mathcal{J}(u)$ with $\lambda_+=0$, and $\lambda_+>0$ respectively. Furthermore, under additional assumption that $F\in C^1([0,+\infty);[0,+\infty))$, local Lipschitz regularity is carried out for non-negative minimizers of $\mathcal{J}(u)$ with $\lambda_+>0$.

Key words: free boundary problem; minimum problem; regularity; growth rate; Orlicz space.

1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^n (n\geq 2)$. Let $g,f\in C([0,\infty);[0,\infty))\cap C^1((0,\infty);(0,\infty))$ with $g(0)=F(0)=0$ satisfy the Lieberman's conditions, which were introduced in [16] for a large class of degenerate/singular elliptic equations, i.e.,

$$0<\delta_0\leq \frac{tg'(t)}{g(t)}\leq g_0,\ \forall\, t>0, \tag{1}$$

with $\delta_0<n-1$, and

$$1+\theta_0\leq \frac{F'(t)t}{F(t)}\leq 1+f_0,\ \forall\, t>0. \tag{2}$$

with θ_0, f_0 satisfying $-1<\theta_0\leq f_0<\delta_0$.

Email addresses: `zhengjun@swjtu.edu.cn` (Jun Zheng [1]), `lean.mat.ufca@gmail.com` (Leandro S. Tavares[2]), `coalves@mat.ufcg.br` (Claudianor O. Alves[3]).

The aim of this paper is to derive interior regularity estimates for a large class of heterogeneous non-differentiable functionals

$$\mathcal{J}(u)=\int_{\Omega}(G(|\nabla u|)+qF(u^{+})+hu+\lambda_{+}\chi_{\{u>0\}})\mathrm{d}x\to \min, \tag{3}$$

among competing functions $u\in\{u\in L^{1}(\Omega):\int_{\Omega}G(|\nabla u|)\mathrm{d}x<\infty, u=\varphi \text{ on } \partial\Omega\}$, where $G(t)=\int_{0}^{t}g(s)\mathrm{d}s$, $\varphi\in L^{\infty}(\Omega)$ with $\int_{\Omega}G(|\nabla\varphi|)\mathrm{d}x<\infty$, $h,q\in L^{\infty}(\Omega)$ with $q\not\equiv 0$, $u^{+}=\max\{u,0\}$, and $\lambda_{+}\geq 0$ is a constant. Note that if no restriction is made on the sign of h, problem (3) has a minimizer that may change its sign near the free boundary $\partial\{u>0\}$.Therefore problem (3) is not in the one-phase case in the strict sense.

A typical form of (3) is the free boundary problem of $p-$Laplacian, i.e.,

$$\int_{\Omega}(|\nabla u|^{p}+q(u^{+})^{\gamma}+hu+\lambda_{+}\chi_{\{u>0\}})\mathrm{d}x\to \min, \tag{4}$$

over the set $\{u\in W^{1,p}(\Omega), u-\varphi\in W_{0}^{1,p}(\Omega)\}$, corresponding to set $g(t)=pt^{p-1},\delta_{0}=g_{0}=p-1,p>1$, and $F(t)=t^{\gamma},\theta_{0}=f_{0}=\gamma-1,0<\gamma<p$ in (1), and (2) respectively. More examples of functions satisfying (1) (or (2)) is given in Remark 2.

A number of important mathematical physics problems, coming from several different contexts, are modeled by optimization setups, for which (4) serves as an emblematic, leading prototype. The case of $\gamma=1,q\not\equiv 0$ and $\lambda_{+}=0$ represents the obstacle type problems. The general case of $0<\gamma<p,q\not\equiv 0$ and $\lambda_{+}=0$ is usually used to model the density of certain chemical specie in reaction with a porous catalyst pellet. The case of $q\equiv 0$ and $\lambda_{+}\neq 0$ relates to jets flow and cavities problems. The minimization problem (4), particularly homogeneous one-phase problem (i.e., $h\equiv 0$, and minimizers of which do not change sign), has indeed received overwhelming attention at aspects of both regularity of solutions and regularity of free boundaries in the past decades, e.g., just to cite a few, [11,13] for the homogeneous one-phase obstacle problems, [3,10,18,19] for the homogeneous one-phase chemical reaction problems with $\gamma\in(0,1)$, [2,9] for the homogeneous one-phase jets flow and cavities problems with $q\equiv 0$ and $\lambda_{+}\neq 0$, and [15] for a large class of homogeneous one-phase free boundary problems of $p-$Laplacian type corresponding to (4) with $h\equiv 0,1\leq\gamma<p$. A rather complete description of regularity theory in the heterogeneous two-phase free boundary problems, which contain (4) with $\gamma\in[0,1]$, and one-phase problems as special cases, was provided in [14].

For the setting in Orlicz spaces, regularities of solutions and free boundaries are addressed for $\lambda_{+}=0,h\equiv 0,q\equiv C$ in [6,7], and for $q\equiv h\equiv 0,\lambda_{+}>0$ in [17], respectively. The homogeneous two-phase jets flow and cavities problems were studied in [5]. It should be mentioned that the heterogeneous two-phase problems related to (3) with $F(t)=t^{\gamma}(\gamma\in(0,1])$, and a version of two-phase problems related to (3) with $F(t)\leq\max\{t^{p},t^{p}\}(p,q\geq 1)$ and $h\equiv 0$, were studied in [20,22], and [4] respectively. Nevertheless, regularities in the problem (3) for a large class of heterogeneous non-differentiable functionals has been little studied in the literature in Orlicz spaces.

The aim of this paper is to consider the free boundary problem (3) and prove several regularities for minimizers of $\mathcal{J}(u)$. Comparing with the existing results, the main contribution of this paper include: (i) our problems concern with not only the non-homogeneous case of $h\not\equiv 0$, but also the case of $F(t)\leq\max\{t^{p},t^{p}\}$ with positive exponents p,q less or larger than 1, which can be seen as complements of [4,5,20,22]; (ii) we establish local Log-Lipschitz continuity for minimizers of $\mathcal{J}(u)$ with $\lambda+>0$ under the assumption that $\delta_{0}>0$, which is weaker than the condition that $\delta_{0}\geq 1$ (or, equivalently, $\frac{g(t)}{t}$ is increasing in t) in [4,22]; (iii) we prove the growth rate near the free boundaries for non-negative minimizers of $\mathcal{J}(u)$ with $h\not\equiv 0$ and F satisfying (2), which is new even in the problem (4) with one-phase and $\gamma\in(0,p)$; (iv) we prove local Lipschitz continuity for non-negative minimizers of $\mathcal{J}(u)$ with $h\not\equiv 0$, which is an extension of [4]; (v) the results obtained in this paper are naturally extensions of the existing results of $p-$Laplican type.

Throughout this paper, without spacial states, we always assume that

$$\begin{aligned}
&g,f\in C([0,\infty);[0,\infty))\cap C^{1}((0,\infty);(0,\infty)), g(0)=F(0)=0,\ g,f \text{ satisfy (1) and (2)},\\
&h,q\in L^{\infty}(\Omega), q\not\equiv 0, \lambda_{+}\geq 0 \text{ is a constant},\\
&\varphi\in W^{1,G}(\Omega)\cap L^{\infty}(\Omega),
\end{aligned}$$

where the definition of $W^{1,G}(\Omega)$ is given in Section 2. For $t>0$, denote by $f(t)$ the derivative of $F(t)$, i.e, $f(t)=F'(t)$, $\forall t>0$. We assume further in Section 4, 5 and 6 that

$$f \text{ is monotone in } t>0.$$

Moreover, we always assume in Section 5 and 6 that there exists $\tau \in (0,1]$ such that

$$\int_t^{t+k} |Q'(s)|\mathrm{d}s \leqslant c_0 \left(\frac{k}{t}\right)^{\tau}, \tag{5}$$

for all $t>0, k>0$, where $Q(s)=\frac{tg'(t)}{g(t)}$, $c_0=c_0(\delta_0,g_0,\tau)$ is a positive constant. Let $\mathcal{K}=\{v\in W^{1,G}(\Omega): v-\varphi\in W_0^{1,G}(\Omega)\}$. Denote a ball in Ω by B, B_r or B_R without special statements on their radius and centres, and denote by $B_r(x_0)$ a ball with radius r and centre x_0. Without confusion, constants $\varepsilon,\tau,c,C,C_0,C_1,...$ appearing in this paper may be different from each other.

The rest of the paper is organized as follows. We provide two remarks on the structural conditions (1) and (2) at the end of Section 1. Some basic properties of functions g, G and F, definitions of Orlicz spaces, properties of functions in Orlicz spaces, and an iteration lemma for the establishment of regularities of minimizers are presented in Section 2. Existence of minimizers (and non-negative minimizers) of $\mathcal{J}(u)$ and their $L^\infty-$boundedness and local $C^{0,\alpha}-$continuity are addressed in Section 3. Local $C^{1,\alpha}-$continuity, and local Log-Lipschitz continuity of minimizers are established in Section 4 for $\mathcal{J}(u)$ with $\lambda+=0$, and $\lambda+>0$ respectively. Growth rate near the free boundary $\partial\{u>0\}$ of each non-negative minimizer of $\mathcal{J}(u)$ with $\lambda+=0$ and $\lambda+>0$ are given in Section 5 respectively. As a consequence of the obtained results, we can prove the optimal growth rates of each non-negative minimizer and its gradient in the one-phase free boundary problems for $p-$Laplacian. Under the further assumption on F, i.e., $F\in C^1([0,+\infty);[0,+\infty))$, Local Lipschitz continuity of non-negative minimizers of $\mathcal{J}(u)$ with $\lambda+>0$ is established in Section 6.

Remark 1 *We do not require any C^2-continuity of F to assume that*

$$\theta_0 \leq \frac{f'(t)t}{f(t)} \leq f_0, \quad \forall\, t>0, \tag{6}$$

provided a C^1-continuos function f satisfying $f(0)=0$ and $F(t)=\int_0^t f(s)ds$. Therefore (2) *is weaker than the structural condition imposed on F by* (6).

Remark 2 *In this remark, we present several functions defined on $[0,+\infty)$ and satisfying condition* (1), *proofs of which and more functions satisfying a slight version of* (1) *can be found in [21].*

(i) $g(t)=(1+t)\ln(1+t)-t$ *satisfies* (1) *with* $\delta_0=1$ *and* $g_0=2$.
(ii) $g(t)=\ln(1+at)+bt$ *satisfies* (1) *with* $\delta_0=\frac{b}{a+b}$ *and* $g_0=\frac{a+b}{b}$, *where* $a>0,b>0$.
(iii) $g(t)=t^a\log_c(bt+d)$ *satisfies* (1) *with* $\delta_0=a$ *and* $g_0=a+\frac{1}{\ln d}$, *where* $,a,b>0,c,d>1$.
(iv) $g(t)=\frac{t^a}{\log_c(bt+d)}$ *satisfies* (1) *with* $\delta_0=a-\frac{1}{\ln d}$ *and* $g_0=a$, *where* $b>0,c,d>1,a>\frac{1}{\ln d}$.
(v) $g(t)=\begin{cases} at^p, 0\leq t<t_0 \\ bt^q+c, t\geq t_0\end{cases}$ *satisfies* (1) *with* $\delta_0=\min\{p,q\}>0$ *and* $g_0=\max\{p,q\}$, *where* $a,b,c,p,q,t_0>0$ *such that* $at_0^p=bt_0^q+c$, *and* $apt_0^{p-1}=bqt_0^{q-1}$.

2 Some auxiliary results

Lemma 1 ([5,17]) *The functions g and G satisfy the following properties:*

(g_1) $\min\{s^{\delta_0},s^{g_0}\}g(t)\leq g(st)\leq \max\{s^{\delta_0},s^{g_0}\}g(t)$, $\forall s,t\geq 0$.
(G_1) *G is convex on $[0,+\infty)$ and C^2- continuous on $(0,+\infty)$.*
(G_2) $\frac{tg(t)}{1+g_0}\leq G(t)\leq tg(t)$, $\forall\, t\geq 0$.
(G_3) $\min\{s^{\delta_0+1},s^{g_0+1}\}\frac{G(t)}{1+g_0}\leq G(st)\leq (1+g_0)\max\{s^{\delta_0+1},s^{g_0+1}\}G(t)$, $\forall t\geq 0$.

(G_4) $G(a+b) \le 2^{g_0}(1+g_0)(G(a)+G(b))$, $\forall\, a, b \ge 0$.
(G_5) $\delta_0 \le \frac{tg_s'(t)}{g_s(t)} \le g_0$ *and* $\frac{1}{1+g_0} \le G_s(1) \le 1$ *for all* $t > 0$*, where* $G_s(t) = \frac{G(st)}{sg(s)}$ *and* $g_s(t) = G_s'(t)$ *for* $s > 0$.

Lemma 2 *The functions F and f satisfy the following properties:*

(F_1) $\min\{s^{1+\theta_0}, s^{1+f_0}\}F(t) \le F(st) \le \max\{s^{1+\theta_0}, s^{1+f_0}\}F(t)$, $\forall s, t \ge 0$.
(F_2) $F(s+t) \le 2^{1+f_0}(F(s)+F(t))$, $\forall s, t \ge 0$.
(f_1) $\frac{1+\theta_0}{1+f_0}\min\{s^{\theta_0}, s^{f_0}\}f(t) \le f(st) \le \frac{1+f_0}{1+\theta_0}\max\{s^{\theta_0}, s^{f_0}\}f(t)$, $\forall s, t > 0$.
(f_2) $\lim\limits_{t\to+\infty}\frac{f(t)}{g(t)} = 0$*, and* $\lim\limits_{t\to+\infty}\frac{F(t)}{G(t)} = 0$.

Proof (F_1) is a consequence of (2) and (g_1).

For (F_2), without loss of generality, assume that $s \le t$, then $F(s+t) \le F(2t) \le 2^{1+f_0}F(t) \le 2^{1+f_0}(F(s)+F(t))$.

For (f_1), we deduce by (2) and (F_1),

$$\begin{aligned} stf(st) \le& (1+f_0)F(st) \\ \le& (1+f_0)\max\{s^{1+\theta_0}, s^{1+f_0}\}F(t) \\ \le& \frac{1+f_0}{1+\theta_0}\max\{s^{1+\theta_0}, s^{1+f_0}\}tf(t), \end{aligned}$$

which yields the second inequality in (f_1). The first inequality in (f_1) can be obtained in a similar way.

For (f_2), we infer from (f_1) and (g_1) that for large $t > 1$,

$$f(t)t^{\delta_0-f_0} \le \frac{1+f_0}{1+\theta_0}f(1)t^{f_0}t^{\delta_0-f_0} \le \frac{1+f_0}{1+\theta_0}\frac{f(1)}{g(1)}g(1)t^{\delta_0} \le \frac{1+\theta_0}{1+f_0}\frac{f(1)}{g(1)}g(t),$$

which implies $\lim\limits_{t\to+\infty}\frac{f(t)}{g(t)} = 0$. The second result can be obtained in a similar way by (F_1) and (G_1). ■

Lemma 3 *The follows statements hold true.*

(F_3) *If $f(t)$ is decreasing in $t > 0$, then* $F(s) - F(t) \le F(s-t)$, $\forall s \ge t \ge 0$.
(F_4) *If $f(t)$ is increasing in $t > 0$, then* $F(s) - F(t) \le f(M)(s-t)$, $\forall 0 \le t \le s \le M$ *with some* $M > 0$.

Proof We prove (F_3). Firstly, note that $F(0) = 0$ due to (F_1). Now Fix $s \ge 0$ and let $v(t) = F(s+t) - F(s) - F(t)$ for any $t \ge 0$. For $t > 0$, we have $v'(t) = f(s+t) - f(t) \le 0$, which yields the nondecreasing monotonicity of v in $t > 0$. By continuity of v in $t = 0$, we conclude that $v(t) \le v(0)$ for all $t \ge 0$, i.e., $F(s+t) \le F(s) + F(t)$. Finally, for $s \ge t \ge 0$, we have $F(s) - F(t) = F(s-t+t) - F(t) \le F(s-t) + F(t) - F(t) = F(s-t)$.

For (F_4), if $s = t = 0$, the result is trivial. If $M \ge s \ge t > 0$, by Mean Value Theorem, it follows $F(s) - F(t) = f(\xi)(s-t)$ with some $\xi \in (t, s) \subset (0, M]$. By the increasing monotonicity of $f(t)$ in $t > 0$, we have $F(s) - F(t) \le f(M)(s-t)$ for all $M \ge s \ge t > 0$. Finally, (f_4) has been proved. ■

As g is strictly increasing we can define its inverse function g^{-1}. Then g^{-1} satisfies a similar condition to (1.2).

Lemma 4 ([17]) *The function g^{-1} satisfies the following property:*

$$\frac{1}{g_0} \le \frac{t(g^{-1})'(t)}{g^{-1}(t)} \le \frac{1}{\delta_0}, \quad \forall\, t > 0.$$

Moreover, g^{-1} *satisfies*

$$(\tilde{g}_1)\quad \min\{s^{\frac{1}{\delta_0}}, s^{\frac{1}{g_0}}\}g^{-1}(t) \le g^{-1}(st) \le \max\{s^{\frac{1}{\delta_0}}, s^{\frac{1}{g_0}}\}g^{-1}(t),$$

and if $\tilde{G}$ *is such that* $\tilde{G}'(t) = g^{-1}(t)$ *then*

$$(\tilde{G}_1)\quad \frac{1+\delta_0}{\delta_0}\min\{s^{1+1/\delta_0}, s^{1+1/g_0}\}\tilde{G}(t) \le \tilde{G}(st) \le \frac{\delta_0}{1+\delta_0}\max\{s^{1+1/\delta_0}, s^{1+1/g_0}\}\tilde{G}(t),$$
$$(\tilde{G}_2)\quad ab \le \varepsilon G(a) + C(\varepsilon)\tilde{G}(b),\ \ \forall\, a,b>0 \text{ and } \varepsilon>0,$$
$$(\tilde{G}_3)\quad \tilde{G}(g(t)) \le g_0 G(t).$$

We recall that the functional

$$\|u\|_{L^G(\Omega)} := \inf\left\{k>0; \int_\Omega G\left(\frac{|u|}{k}\right)\mathrm{d}x \le 1\right\}$$

is a norm in the Orlicz space $L^G(\Omega)$ which is the linear hull of the Orlicz class

$$\mathcal{K}_G(\Omega) := \left\{u\ \text{ measurable}; \int_\Omega G(|u|)\mathrm{d}x < +\infty\right\}.$$

Notice that this set is convex, since G is also convex. The Orlicz-Sobolev space $W^{1,G}(\Omega)$ is defined as

$$W^{1,G}(\Omega) := \{u \in L^G(\Omega); \nabla u \in (L^G(\Omega))^n\},$$

which is the usual subspace of $W^{1,1}(\Omega)$, and associated with the norm

$$\|u\|_{W^{1,G}(\Omega)} = \|u\|_{L^G(\Omega)} + \|\nabla u\|_{L^G(\Omega)}.$$

We present some properties of spaces $L^G(\Omega)$ and $W^{1,G}(\Omega)$, and properties of functions in $L^G(\Omega)$ and $W^{1,G}(\Omega)$.

Lemma 5 ([17]) *There exists a constant* $C = C(\delta_0, g_0)$ *such that*

$$\|u\|_{L^G(\Omega)} \le C\max\left\{\left(\int_\Omega G(|u|)dx\right)^{\frac{1}{1+\delta_0}}, \left(\int_\Omega G(|u|)dx\right)^{\frac{1}{1+g_0}}\right\}.$$

Lemma 6 ([17]) $L^{\tilde{G}}(\Omega)$ *is the dual of* $L^G(\Omega)$*. Moreover,* $L^G(\Omega)$ *and* $W^{1,G}(\Omega)$ *are reflexive.*

Lemma 7 ([17]) $L^G(\Omega) \hookrightarrow L^{1+\delta_0}(\Omega)$ *continuously.*

Lemma 8 ([17]) *For any* $u \in L^G(\Omega)$ *and any* $v \in L^{\tilde{G}}(\Omega)$*, there holds* $\left|\int_\Omega uv dx\right| \le 2\|u\|_{L^G(\Omega)}\|v\|_{L^{\tilde{G}}(\Omega)}$.

Lemma 9 ([8]) *For any* $u \in W_0^{1,G}(\Omega)$*, which is the closure of* $C_0^\infty(\Omega)$ *in* $W^{1,G}(\Omega)$*, there holds* $\int_\Omega G(|u|)dx \le \int_\Omega G(c|\nabla u|)dx$*, where the constant* c *is twice the diameter of* Ω.

Lemma 10 ([17]) *Let* $u \in L^\infty(\Omega)$ *such that for some* $\alpha \in (0,1)$ *and* $r_0 > 0$,

$$\int_{B_r} G(|\nabla u|)dx \le C_1 r^{n+\alpha-1},\ \ 0 < r \le r_0,$$

with $B_{r_0} \Subset \Omega$*. Then* $u \in C^\alpha(\Omega)$ *and there exists a constant* $C = C(C_1, \alpha, n, g_0, G(1))$ *such that* $[u]_{0,\alpha,\Omega} \le C$.

Lemma 11 *Let v be a bounded weak solution of div $\frac{g(|\nabla v|)}{|\nabla v|}\nabla v = 0$ in B_R (see (54) for a definition). For every $\lambda \in (0, n)$, there exists $C = C(\lambda, n, \delta, g_0, \|v\|_{L^\infty(B_R)}) > 0$ such that*

$$\int_{B_r} G(|\nabla v|)dx \leq Cr^\lambda, \quad \forall 0 < r \leq R.$$

Proof See [16, (5.9), page 346], or [17, Lemma 2.7]. ■

Lemma 12 ([20]) *Let $u \in W^{1,G}(\Omega)$, $B_R \subset \Omega$. If v is a bounded weak solution of*

$$div\ \frac{g(|\nabla v|)}{|\nabla v|}\nabla v = 0 \quad in\ B_R, \quad v - u \in W_0^{1,G}(B_R),$$

then for any $\lambda \in (0, n)$, there exists $C = C(\lambda, n, \delta_0, g_0, \|v\|_{L^\infty(B_R)}) > 0$ such that

$$\int_{B_R} G(|\nabla u - \nabla v|)dx \leq C\int_{B_R} (G(|\nabla u|) - G(|\nabla v|))dx + CR^{\frac{\lambda}{2}}\left(\int_{B_R} (G(|\nabla u|) - G(|\nabla v|))dx\right)^{\frac{1}{2}}.$$

Proof *See the proof of Lemma 3.1 in [20].* ■

Let $(u)_r = \frac{1}{|B_r|}\int_{B_r} u\mathrm{d}x$ be the average value of function u on the ball B_r.

Lemma 13 ([20]) *Let $u \in W^{1,G}(\Omega)$, $B_R \subset \Omega$. If $v \in W^{1,G}(B_R)$ is a weak solution of div $\frac{g(|\nabla v|)}{|\nabla v|}\nabla v = 0$ in B_R, then for some positive constant $0 < \sigma < 1$, there exists a positive constant $C = C(n, \delta, g_0)$ such that for each $0 < r \leq R$, there holds*

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|)dx \leq C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u - (\nabla u)_R|)dx + C\int_{B_R} G(|\nabla u - \nabla v|)dx.$$

The following result is an iteration lemma, which will be used in the establishment of regularities of minimizers of $\mathcal{J}$.

Lemma 14 ([14]) *Let $\overline{\phi}(s)$ be a non-negative and non-decreasing function. Suppose that*

$$\overline{\phi}(r) \leq C_1\left(\left(\frac{r}{R}\right)^\alpha + \vartheta\right)\overline{\phi}(R) + C_2R^\beta,$$

for all $r \leq R \leq R_0$, with C_1, α, β positive constants and C_2, ϑ non-negative constants. Then, for any $\tau < \min\{\alpha, \beta\}$, there exists a constant $\vartheta_0 = \vartheta_0(C_1, \alpha, \beta, \tau)$ such that if $\vartheta < C_1, \vartheta_0$, then for all $r \leq R \leq R_0$ we have

$$\overline{\phi}(r) \leq C_3\left(\frac{r}{R}\right)^\tau(\overline{\phi}(R) + C_2R^\tau),$$

where $C_3 = C_3(C_1, \tau - \min\{\alpha, \beta\})$ is a positive constant. In turn,

$$\overline{\phi}(r) \leq C_4r^\tau,$$

where $C_4 = C_4(C_2, C_3, R_0, \overline{\phi}, \tau)$ is a positive constant.

3 Existence, $L^\infty-$boundedness and continuity of minimizers over the set $\mathcal{K}$

Theorem 15 (Existence and $L^\infty-$boundedness) *There exists a minimizer u of the functional $\mathcal{J}(u)$ over the set $\mathcal{K}$, and there exists a constant $C_0 > 0$ depending only on $\delta_0, g_0, \lambda_+, G(1), \tilde{G}(1)$, and $\|\varphi\|_{L^\infty(\Omega)}$ such that*

$$\|u\|_{L^\infty(\Omega)} \leq C_0,$$

for all minimizers u of $\mathcal{J}(u)$ over the set $\mathcal{K}$.

Proof We will prove that

$$I_0 := \inf\{\mathcal{J}(v); v \in \mathcal{K}\} > -\infty.$$

For $v \in \mathcal{K}$, we have from Lemma 9 that

$$\int_\Omega G(|v-\varphi|)\mathrm{d}x \leq \int_\Omega G(c|\nabla v - \nabla\varphi|)\mathrm{d}x, c = 2\mathrm{diam}(\Omega). \tag{7}$$

Thus, using $(G_3), (G_4)$ and the nondecreasing monotonicity of G we have

$$\begin{aligned}\int_\Omega G(|v|)\mathrm{d}x \leq &\int_\Omega G(|v-\varphi|+|\varphi|)\mathrm{d}x\\ \leq &C\left(\int_\Omega G(|v-\varphi|)\mathrm{d}x + \int_\Omega G(|\varphi|)\mathrm{d}x\right)\\ \leq &C\left(\int_\Omega G(|\nabla v-\nabla\varphi|)\mathrm{d}x + \int_\Omega G(|\varphi|)\mathrm{d}x\right)\\ \leq &C\left(\int_\Omega G(|\nabla v|\mathrm{d}x + \int_\Omega G(\nabla\varphi|)\mathrm{d}x + \int_\Omega G(|\varphi|)\mathrm{d}x\right),\end{aligned} \tag{8}$$

where C is constant depending only on the diameter of Ω, and δ_0 and g_0.

(f_2) implies that for given $\tau > 0$, there exists a constant $K_\tau > 0$ such that

$$F(t) \leq K_\tau + \tau G(t), \quad \forall t \geq 0. \tag{9}$$

By (9), we have

$$\left|\int_\Omega qF(v^+)\mathrm{d}x\right| \leq \|q\|_{L^\infty(\Omega)}\int_\Omega F(v^+)\mathrm{d}x \leq \|q\|_{L^\infty(\Omega)}K_\tau + \|q\|_{L^\infty(\Omega)}\tau\int_\Omega G(|v|)\mathrm{d}x.$$

From $(\tilde{G}_2)$ we have for all $\tau > 0$ that

$$\left|\int_\Omega hv\mathrm{d}x\right| \leq \tau\int_\Omega G(|v|)\mathrm{d}x + C_\tau\int_\Omega \tilde{G}(|h|)\mathrm{d}x,$$

where $C_\tau > 0$ is a constant independent of v.

Therefore

$$\left|\int_\Omega qF(v^+)\mathrm{d}x\right| + \left|\int_\Omega hv\mathrm{d}x\right| \leq \tau C\left(\int_\Omega G(|\nabla v|\mathrm{d}x + G(|\nabla\varphi|) + G(|\varphi|)\mathrm{d}x\right) + C, \tag{10}$$

where $C > 0$ is a constant independent of v. A convenient choice of $\tau > 0$ in (10) implies that

$$\int_\Omega G(|\nabla v|)\mathrm{d}x - \left|\int_\Omega qF(v^+)\mathrm{d}x\right| - \left|\int_\Omega hv\mathrm{d}x\right| + \int_\Omega \lambda\chi_{\{v>0\}} \geq K\left(1+\int_\Omega G(|\nabla\varphi|)+G(|v|)\mathrm{d}x\right),$$

with some positive constant K, which implies that $I_0 > -\infty$.

Now consider $v_j (j \in \mathbb{N})$, a minimizing sequence, and $j_0 \in \mathbb{N}$, such that $\mathcal{J}(v_j) \leq I_0 + 1$ for all $j \geq j_0$. We have for $\tau > 0$ that

$$\begin{aligned}\int_\Omega G(|\nabla v_j|)\mathrm{d}x =&\mathcal{J}(v_j) - \int_\Omega qF(v^+)\mathrm{d}x - \lambda_+\chi_{\{v>0\}}\mathrm{d}x \\ \leq&\tau C\left(\int_\Omega G(|\nabla v_j|)\mathrm{d}x + G(|\nabla\varphi|) + G(|\varphi|)\mathrm{d}x\right) + K,\end{aligned} \tag{11}$$

for all $j \geq j_0$, where $C, K > 0$ are constants independent of $j \in \mathbb{N}$.

A convenient choice of τ in (11) implies that

$$\int_\Omega G(|\nabla v_j|)\mathrm{d}x \leq C + C\left(\int_\Omega G(|\nabla\varphi|)+G(|\varphi|)\mathrm{d}x\right), \quad j \geq j_0, \tag{12}$$

where $C > 0$ is a constant that does not depend on $j \in \mathbb{N}$.

From (G_4), (12) and Lemma 9, we deduce that $v_j - \varphi$ is a bounded sequence in $W_0^{1,G}(\Omega)$. Since $W_0^{1,G}(\Omega)$ is reflexive, there exists $u \in W^{1,G}(\Omega)$ with $u - \varphi \in W_0^{1,G}(\Omega)$ such that for a subsequence we have

$$v_j \rightharpoonup u \text{ in } W^{1,G}(\Omega).$$

Then by Lemma 7, we find that

$$v_j \rightharpoonup u \ \text{ in } W^{1,1+\delta_0}(\Omega).$$

Thus, up to a subsequence, we have that

$$v_j \to u \text{ in } L^{1+\delta_0}(\Omega), v_j \to u \text{ a.e in } \Omega,\ u = \varphi \text{ on } \partial\Omega,\ \text{and } |v_j| \leq h \text{ a.e in } \Omega, \tag{13}$$

for some $h \in L^{1+\delta_0}(\Omega)$.

Note that

$$\int_\Omega G(|\nabla u|)\mathrm{d}x \leq \liminf_{j\to\infty}\int_\Omega G(|\nabla v_j|)\mathrm{d}x. \tag{14}$$

In fact, by the convexity of G, it follows

$$\int_\Omega G(|\nabla v_j|)\mathrm{d}x \geq \int_\Omega G(|\nabla u|)\mathrm{d}x + \int_\Omega g(|\nabla u|)\frac{\nabla u}{|\nabla u|}(\nabla v_j - \nabla u)\mathrm{d}x. \tag{15}$$

We have that

$$\tilde{G}\left(g(|\nabla u|)\frac{\partial u}{\partial x_i}\frac{1}{|\nabla u|}\right) \leq \tilde{G}(g(|\nabla u|)) \leq CG(|\nabla u|),$$

which implies that $g(|\nabla u|)\frac{\nabla u}{|\nabla u|} \in (L^{\tilde{G}}(\Omega))^n$. Thus, combining the fact that $\nabla v_j \rightharpoonup \nabla u$ in $(L^G(\Omega))^n$ with the inequality (15), we get (14). Using Lemma 2, (13) and the Lebesgue's Dominated Convergence Theorem, we have

$$\int_\Omega F((v_j)^+)\mathrm{d}x \to \int_\Omega qF(v^+)\mathrm{d}x. \tag{16}$$

Using again the Lebesgue's Dominated Convergence Theorem, we get

$$\int_\Omega \lambda_+\chi_{\{v_j>0\}}\mathrm{d}x \to \int_\Omega \lambda_+\chi_{\{u>0\}}\mathrm{d}x. \tag{17}$$

Since $v_j \to u$ in $L^{1+\delta_0}(\Omega)$ and $h \in L^\infty(\Omega)$, we have

$$\int_\Omega |h||v_j - u|\mathrm{d}x \to 0. \tag{18}$$

Thus from (14), (16), (17) and (18) we deduce that $\mathcal{J}(u) \leq \liminf_{j\to+\infty} \mathcal{J}(v_j)$, which implies that u is a minimizer of $\mathcal{J}(u)$.

Now we prove the boundedness of minimizers. Let $j_0 \in \mathbb{N}$ such that $j_0 \geq \|\varphi\|_{L^\infty(\Omega)}$. For each $j \geq j_0$, define the function $u_j : \Omega \to \mathbb{R}$ by

$$u_j = \begin{cases} j \cdot sgn(u) & \text{if } |u| > j, \\ u & \text{if } |u| \leq j, \end{cases}$$

where $sgn(u) = 1$ if $u \geq 0$, and $sgn(u) = -1$ if $u < 0$. Define the set $A_j := \{|u| > j\}$. For each $j \geq j_0$, we have

$$u = u_j \text{ in } {A_j}^c, \quad \text{and} \quad u_j = j \cdot sgn(u) \text{ in } A_j.$$

We have $\{|u_j| > 0\} = \{|u| > 0\}$ for all $j \in \mathbb{N}$, and $(|u| - j)^+ \in W_0^{1,G}(\Omega)$ for all $j \geq j_0$. Note that

$$\begin{aligned}\int_{A_j} G(|\nabla u|)\mathrm{d}x &= \int_\Omega G(|\nabla u|) - G(|\nabla u_j|)\mathrm{d}x \\ &\leq \int_{A_j} q(F(u_j^+) - F(u^+)) + \lambda_+(\chi_{\{u_j>0\}} - \chi_{\{u>0\}}) + h(u_j - u)\mathrm{d}x \\ &\leq \|q\|_{L^\infty(\Omega)} \int_{A_j} |F(u_j^+) - F(u^+)|\mathrm{d}x + \int_{A_j} h(u_j - u)\mathrm{d}x. \end{aligned} \tag{19}$$

By (F_2), we have

$$\begin{aligned}\int_{A_j} |F(u_j^+) - F(u^+)|\mathrm{d}x &= \int_{A_j\cap\{u>0\}} F(u) - F(j)\mathrm{d}x \\ &= \int_{A_j\cap\{u>0\}} F(u - j + j) - F(j)\mathrm{d}x \\ &\leq \int_{A_j\cap\{u>0\}} C(F(u - j) + F(j))\mathrm{d}x \\ &\leq C\int_{A_j\cap\{u>0\}} F(u - j)\mathrm{d}x + CF(j)|A_j|. \end{aligned} \tag{20}$$

In view of (9) and using Lemma 9, we have

$$\int_{A_j\cap\{u>0\}} F(u - j)\mathrm{d}x \leq C_\tau|A_j| + \tau\int_{A_j\cap\{u>0\}} G(u - j)\mathrm{d}x$$

$$\begin{aligned}&\le C_\tau|A_j|+\tau\int_{A_j}G((|u|-j)^+)\mathrm{d}x\\&\le C_\tau|A_j|+\tau C\int_{A_j}G(|\nabla u|)\mathrm{d}x.\end{aligned}\tag{21}$$

By (19), (20), (21) and a suitable choice of $\tau>0$, we get

$$\int_{A_j}G(|\nabla u|)\mathrm{d}x\le C|A_j|+CF(j)|A_j|+\int_{A_j}h(u_j-u)\mathrm{d}x\tag{22}$$

Arguing as in [22, (3.12),(3.13)], we have for all $\varepsilon>0$ that

$$\begin{aligned}\int_{A_j}h(u_j-u)\mathrm{d}x\le&2\int_{A_j}|h|(|u|-j)\mathrm{d}x\\\le&C\left(\varepsilon\int_{A_j}G(|\nabla u|)\mathrm{d}x+|A_j|^{1+\frac{1}{n}\frac{1+g_0}{g_0}}\right),\end{aligned}\tag{23}$$

where C is a constant that does not depend on ε. Using (22) and considering a convenient choice of ε in (23) we get

$$\int_{A_j}G(|\nabla u|)\mathrm{d}x\le C|A_j|+CF(j)|A_j|+C|A_j|^{1+\frac{1}{n}\frac{1+g_0}{g_0}}.$$

Without lost of generality we only consider that $|A_j|\le 1$ for all $j\ge j_0$. Otherwise we may replace $|A_j|$ with $\frac{|A_j|}{|\Omega|}$. Therefore for $j_0\in\mathbb{N}$ large enough, we have

$$\int_{A_j}G(|\nabla u|)\mathrm{d}x\le CF(j)|A_j|,\forall j\ge j_0.\tag{24}$$

By (G_3), we have

$$\int_{A_j\cap\{|\nabla u|<1\}}|\nabla u|^{1+g_0}\mathrm{d}x\le\frac{1+g_0}{G(1)}\int_{A_j\cap\{|\nabla u|<1\}}G(|\nabla u|)\mathrm{d}x,\tag{25}$$

$$\int_{A_j\cap\{|\nabla u|\ge1\}}|\nabla u|^{1+\delta_0}\mathrm{d}x\le\frac{1+g_0}{G(1)}\int_{A_j\cap\{|\nabla u|\ge1\}}G(|\nabla u|)\mathrm{d}x.\tag{26}$$

By Hölder's inequality, we get

$$\begin{aligned}\int_{A_j\cap\{|\nabla u|<1\}}|\nabla u|^{1+\delta_0}\mathrm{d}x\le&|A_j\cap\{|\nabla u|<1\}|^{\frac{g_0-\delta_0}{1+g_0}}\left(\int_{A_j\cap\{|\nabla u|<1\}}|\nabla u|^{1+g_0}\right)^{\frac{1+\delta_0}{1+g_0}}\\\le&|A_j|^{\frac{g_0-\delta_0}{1+g_0}}\left(\int_{A_j\cap\{|\nabla u|<1\}}|\nabla u|^{1+g_0}\right)^{\frac{1+\delta_0}{1+g_0}}.\end{aligned}\tag{27}$$

By (24), (25), (26), (27) and Lemma 1, we have

$$\begin{aligned}\int_{A_j}|\nabla u|^{1+\delta_0}\mathrm{d}x=&\int_{A_j\cap\{|\nabla u|<1\}}|\nabla u|^{1+\delta_0}\mathrm{d}x+\int_{A_j\cap\{|\nabla u|\ge1\}}|\nabla u|^{1+\delta_0}\mathrm{d}x\\\le&C|A_j|^{\frac{g_0-\delta_0}{1+g_0}}(F(j))|A_j|)^{\frac{1+\delta_0}{1+g_0}}+F(j)|A_j|\\\le&C(F(1))^{\frac{1+\delta_0}{1+g_0}}j^{(1+f_0)\frac{1+\delta_0}{1+g_0}}|A_j|+CF(1)j^{1+f_0}|A_j|\\\le&C(1+G(1))^{\frac{1+\delta_0}{1+g_0}}j^{(1+f_0)\frac{1+\delta_0}{1+g_0}}|A_j|+C(1+G(1))j^{1+f_0}|A_j|,\end{aligned}$$

where in the last inequality we used (f_2) and the monotonicity of F such that the constant depends on $G(1)$, rather than $F(1)$. Since $0 < f_0 < \delta_0 < g_0$, it follows

$$\int_{A_j} |\nabla u|^{1+\delta_0} \mathrm{d}x \leq Cj^{1+\delta_0}|A_j| = Cj^{1+\delta_0}|A_j|^{1-\frac{1+\delta_0}{n}+\frac{1+\delta_0}{n}}. \tag{28}$$

where C is a constant that is independent of $j \in \mathbb{N}$. By Lemma 8, we have

$$\int_{A_{j_0}} |u| \mathrm{d}x \leq |A_{j_0}|^{\frac{\delta_0}{1+\delta_0}} \|u\|_{L^{1+\delta_0}(A_{j_0})} \leq C. \tag{29}$$

Then by (28), (29) and [12, Lemma 5.2, Chap. 2], we obtain the boundedness of minimizers. ■

Remark 3 *Note that from the proof of Theorem 15, we conclude that minimizers of $\mathcal{J}(u)$ over the set $\mathcal{K}$ are bounded in $W^{1,G}(\Omega)$. Indeed, arguing as in (8), we have*

$$\int_\Omega G(|u|)dx \leq C \int_\Omega \left(G(|\nabla u|) + G(|\nabla \varphi|) + G(|\varphi|)\right) dx.$$

For $\varepsilon > 0$, we have

$$\begin{aligned}\int_\Omega G(|\nabla u|)dx =& \mathcal{J}(u) - \int_\Omega (qF(u^+) + \lambda_+ \chi_{\{u>0\}})dx \\ \leq& \mathcal{J}(u) - \int_\Omega qF(u^+)dx \\ \leq& \mathcal{J}(\varphi) + \|q\|_{L^\infty(\Omega)} \int_\Omega F(|u|)dx \\ \leq& \mathcal{J}(\varphi) + \|q\|_{L^\infty(\Omega)} \left(C_\varepsilon + \varepsilon \int_\Omega G(|\nabla u|)dx\right),\end{aligned}$$

where C_ε is a constant that depends only on ε. A suitable choice of $\varepsilon > 0$ implies that

$$\int_\Omega G(|\nabla u|)dx \leq C\mathcal{J}(\varphi) + C,$$

where C is a constant that is independent of u. Then

$$\int_\Omega G(|u|)dx \leq C.$$

Finally, we conclude that minimizers of $\mathcal{J}(u)$ are bounded in $W^{1,G}(\Omega)$.

Theorem 16 (Local $C^{0,\alpha}$−continuity) *Let u be a minimizer of $\mathcal{J}(u)$ over the set $\mathcal{K}$. Then $u \in C^{0,\alpha}_{loc}(\Omega)$ for all $\alpha \in (0,1)$.*

Proof For any ball $B_R \Subset \Omega$, let v be a weak solution of the following equation

$$\begin{cases} \operatorname{div} \frac{g(|\nabla v|)}{|\nabla v|}\nabla v = 0 & \text{in } B_R, \\ \qquad\qquad v = u & \text{on } \partial B_R. \end{cases}$$

By the minimality of u, we have

$$\int_{B_R} G(|\nabla u|)\mathrm{d}x - \int_{B_R} G(|\nabla v|)\mathrm{d}x \leq \int_{B_R} q(F(v^+) - F(u^+))\mathrm{d}x + \lambda_+ \int_{B_R} (\chi_{\{v>0\}} - \chi_{\{u>0\}})\mathrm{d}x$$

$$
\begin{aligned}
&\leq \|q\|_{L^\infty(B_R)} \cdot \int_{B_R} (|F(v^+)| + |F(u^+)|)\mathrm{d}x + \lambda_+ R^n \\
&\leq \|q\|_{L^\infty(\Omega)} \cdot \int_{B_R} (F(\|v\|_{L^\infty(B_R)}) + F(\|u\|_{L^\infty(B_R)}))\mathrm{d}x + \lambda_+ R^n \\
&\leq CR^n,
\end{aligned} \tag{30}
$$

where we used the increasing property of F, and the fact that $\|v\|_{L^\infty(B_R)} \leq \|v\|_{L^\infty(\partial B_R)} = \|u\|_{L^\infty(\partial B_R)} \leq \|u\|_{L^\infty(B_R)} \leq \|u\|_{L^\infty(\Omega)} \leq C$, which is guaranteed by the maximum principle. By (30) and Lemma 11, for any $\lambda \in (0, n)$, there holds

$$
\int_{B_R} G(|\nabla u|)\mathrm{d}x \leq CR^n + \int_{B_R} G(|\nabla v|)\mathrm{d}x \leq CR^n + CR^\lambda \leq CR^\lambda.
$$

Particulary, we have

$$
\int_{B_R} G(|\nabla u|)\mathrm{d}x \leq CR^{n+\alpha-1}.
$$

for any $\alpha \in (0, 1)$. We conclude the desired result by Lemma 10. ∎

Corollary 17 *Assume further that $h \leq 0$ a.e. in Ω, and $\varphi \geq 0$ on $\partial\Omega$ in the sense of trace. Then every minimizer of $\mathcal{J}(u)$ over the set $\mathcal{K}$ is non-negative in Ω.*

Proof Let $\xi = \min\{u, 0\} \leq 0$ and $\varepsilon \in (0, 1)$. By the minimality of u, it follows

$$
\int_\Omega \Big(G(|\nabla(u - \varepsilon\xi)|) - G(|\nabla u|) + q\big(F((u - \varepsilon\xi)^+) - F(u^+)\big) - h\varepsilon\xi + \lambda_+(\chi_{\{u - \varepsilon\xi > 0\}} - \chi_{\{u>0\}}) \Big)\mathrm{d}x \geq 0.
$$

Note that

$$
\begin{aligned}
\int_\Omega q\big(F((u - \varepsilon\xi)^+) - F(u^+)\big)\mathrm{d}x &= \int_{\{u>0\}} q\big(F((u - \varepsilon\xi)^+) - F(u^+)\big)\mathrm{d}x + \int_{\{u\leq 0\}} q\big(F((u - \varepsilon\xi)^+) - F(u^+)\big)\mathrm{d}x \\
&= \int_{\{u>0\}} q\big(F(u^+) - F(u^+)\big)\mathrm{d}x + \int_{\{u\leq 0\}} q\big(F((1-\varepsilon)u^+) - F(u^+)\big)\mathrm{d}x \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_\Omega \lambda_+(\chi_{\{u-\varepsilon\xi>0\}} - \chi_{\{u>0\}})\mathrm{d}x &= \int_{\{u>0\}} \lambda_+(\chi_{\{u>0\}} - \chi_{\{u>0\}})\mathrm{d}x + \int_{\{u\leq 0\}} \lambda_+(\chi_{\{(1-\varepsilon)u>0\}} - \chi_{\{u>0\}})\mathrm{d}x \\
&= 0.
\end{aligned}
$$

Therefore

$$
0 \leq \frac{1}{\varepsilon} \int_\Omega \Big(G(|\nabla(u - \varepsilon\xi)|) - G(|\nabla u|) + q(F((u - \varepsilon\xi)^+) - F(u^+)) - h\varepsilon\xi + \lambda_+(\chi_{\{u-\varepsilon\xi>0\}} - \chi_{\{u>0\}}) \Big)\mathrm{d}x \tag{31}
$$

$$
\leq - \int_\Omega g(|\nabla u - \varepsilon\nabla\xi|) \frac{\nabla u - \varepsilon\nabla\xi}{|\nabla u - \varepsilon\nabla\xi|} \nabla\xi \mathrm{d}x, \tag{32}
$$

where we used the convexity of G and $h\xi \geq 0$ a.e. in Ω. Letting $\varepsilon \to 0^+$, we get $\int_\Omega \frac{g(|\nabla u|)}{|\nabla u|} \nabla u \nabla\xi \mathrm{d}x \geq 0$. It follows

$$
\int_{\{u<0\}} g(|\nabla u|)|\nabla u|\mathrm{d}x \leq 0,
$$

which implies $u \equiv C$ or $u \geq 0$ a.e. in Ω. By the fact that $u = \varphi \geq 0$ on $\partial\Omega$, and the continuity of u, we conclude that $u \geq 0$ in Ω. ∎

Remark 4 *Without restrictions on the sign of* h*, it is easy to see by checking the proof of Theorem 15 and 16 that there exists a non-negative minimizer, which is also bounded and* $C^{0,\alpha}_{loc}-$*contunous, of the functional in* (3) *over the set* $\widetilde{\mathcal{K}} = \{v \in W^{1,G}(\Omega) : v - \varphi \in W^{1,G}_0(\Omega), v \geq 0 \; a.e.in \; \Omega\}$ *provided a non-negative* φ*.*

4 Local $C^{1,\alpha}-$ and Log-Lipschitz regularities of minimizers over the set $\mathcal{K}$

In this section, we establish local $C^{1,\alpha}-$ and Log-Lipschitz continuities for minimizers of $\mathcal{J}$ with $\lambda_+ = 0$ and $\lambda_+ > 0$ respectively.

Theorem 18 (Local $C^{1,\alpha}-$regularity of minimizers for $\lambda_+ = 0$) *Let* u *be a minimizer of* $\mathcal{J}(u)$ *over the set* $\mathcal{K}$ *with* $\lambda_+ = 0$*. Then* $u \in C^{1,\alpha}_{loc}(\Omega)$ *for some* $\alpha \in (0,1)$*. More precisely, for any* $\Omega' \Subset \Omega$*, there exists a constant* $C > 0$*, depending only on* $n, \theta_0, f_0, \delta_0, g_0, G(1), \frac{1}{G(1)}, \|u\|_{L^\infty(\Omega)}, \|h\|_{L^\infty(\Omega)}, \|q\|_{L^\infty(\Omega)}, \|\varphi\|_{L^\infty(\Omega)}$ *and* Ω'*, such that*

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C.$$

Theorem 19 (Local Log-Lipschitz regularity of minimizers for $\lambda_+ > 0$) *Let* u *be a minimizer of* $\mathcal{J}(u)$ *over the set* $\mathcal{K}$ *with* $\lambda_+ > 0$*. Then* u *is locally Log-Lipschitz continuous. More precisely, for any* $\Omega' \Subset \Omega$*, there exists a constant* $C > 0$*, depending only on* $n, \theta_0, f_0, \delta_0, g_0, G(1), \frac{1}{G(1)}, \|u\|_{L^\infty(\Omega)}, \|h\|_{L^\infty(\Omega)}, \|q\|_{L^\infty(\Omega)}, \|\varphi\|_{L^\infty(\Omega)}$ *and* Ω'*, such that*

$$|u(x) - u(y)| \leq C|x-y||\log|x-y||,$$

for any $x, y \in \Omega'$*. Therefore,* $u \in C^{0,\tau}_{loc}(\Omega)$ *for any* $\tau < 1$*.*

Proof of Theorem 18 Let $B_R = B_R(x_0)$ for some $R \leq R_0 \leq 1$, where R_0 will be chosen later. Without loss of generality, assume that $B_r \Subset B_R \Subset \Omega$, and B_r and B_R have the same centre. Let v be a $G-$harmonic function in B_R that agrees with u on the boundary, i.e.,

$$\operatorname{div} \frac{g(|\nabla v|)}{|\nabla v|}\nabla v = 0 \text{ in } B_R \text{ and } v - u \in W^{1,G}_0(B_R).$$

By Lemma 13 and Lemma 12, we have

$$\begin{aligned}\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \leq & C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u - (\nabla u)_R|)\mathrm{d}x + C\int_{B_R} G(|\nabla u - \nabla v|)\mathrm{d}x \\ \leq & C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u - (\nabla u)_R|)\mathrm{d}x + C\int_{B_R} (G(|\nabla u|) - G(|\nabla v|))\mathrm{d}x \\ & + CR^{\frac{\lambda}{2}}\left(\int_{B_R} (G(|\nabla u|) - G(|\nabla v|))\mathrm{d}x\right)^{\frac{1}{2}},\end{aligned} \tag{33}$$

where λ is an arbitrary constant in $(0, n)$.

The minimality of u and the increasing monotonicity of F imply that

$$\begin{aligned}\int_{B_R} (G(|\nabla u|) - G(|\nabla v|))\mathrm{d}x \leq & \int_{B_R} \Big(q(F(v^+) - F(u^+)) + h(v-u)\Big)\mathrm{d}x. \\ \leq & \|q\|_{L^\infty(B_R)}\int_{B_R} |F(v^+) - F(u^+)|\mathrm{d}x + \|h\|_{L^\infty(B_R)}\int_{B_R} |v-u|\mathrm{d}x.\end{aligned} \tag{34}$$

If f in decreasing in $t > 0$, we infer from (F_3) and (F_1) that

$$\int_{B_R} |F(v^+) - F(u^+)|\mathrm{d}x = \int_{B_R \cap \{v^+ \geq u^+\}} (F(v^+) - F(u^+))\mathrm{d}x + \int_{B_R \cap \{v^+ < u^+\}} (F(u^+) - F(v^+))\mathrm{d}x$$

$$
\begin{aligned}
&\leq \int_{B_R\cap\{v^+\geq u^+\}} F(v^+-u^+)\mathrm{d}x + \int_{B_R\cap\{v^+<u^+\}} F(u^+-v^+)\mathrm{d}x \\
&\leq F(1)\max\left\{\int_{B_R}|v-u|^{1+\theta_0}\mathrm{d}x, \int_{B_R}|v-u|^{1+f_0}\mathrm{d}x\right\} \\
&= F(1)\int_{B_R}|v-u|^{1+\theta_0}\mathrm{d}x \\
&\leq C(1+G(1))\int_{B_R}|v-u|^{1+\theta_0}\mathrm{d}x
\end{aligned} \tag{35}
$$

where without loss of generality we assume that $\|v-u\|_{L^\infty(B_R)}\leq 1$ due to the boundedness of v and u.

A similar argument as in [20, (15) on page 44] gives

$$
\int_{B_R}|v-u|^{1+\theta_0}\mathrm{d}x \leq C(\varepsilon)R^{n+\alpha_0}+\varepsilon R^{\beta_0}\int_{B_R}G(|\nabla v-\nabla u|)\mathrm{d}x, \tag{36}
$$

where $\alpha_0,\beta_0>0$ are independent of R, ε_0 will be chosen later.

If f in increasing in $t>0$, we infer from (F_4), (f_1), (f_2) and the boundedness of v and u that

$$
\begin{aligned}
\int_{B_R}|F(v^+)-F(u^+)|\mathrm{d}x &= \int_{B_R\cap\{v^+=u^+=0\}}|F(v^+)-F(u^+)|\mathrm{d}x + \int_{B_R\cap\{v^++u^+\neq 0\}}|F(v^+)-F(u^+)|\mathrm{d}x \\
&\leq \int_{B_R\cap\{v^++u^+\neq 0\}} f(\xi)|v^+-u^+|\mathrm{d}x \\
&\leq \int_{B_R\cap\{v^++u^+\neq 0\}} f(\|\xi\|_{L^\infty(B_R)})|v^+-u^+|\mathrm{d}x \\
&\leq \int_{B_R\cap\{v^++u^+\neq 0\}} f(\|u\|_{L^\infty(\Omega)})|v^+-u^+|\mathrm{d}x \\
&\leq \int_{B_R\cap\{v^++u^+\neq 0\}} \frac{1+f_0}{1+\theta_0}\max\{\|u\|^{\theta_0}_{L^\infty(\Omega)},\|u\|^{f_0}_{L^\infty(\Omega)}\}f(1)|v^+-u^+|\mathrm{d}x \\
&\leq C(1+g(1))\int_{B_R}|v-u|\mathrm{d}x \\
&\leq C(1+G(1))\int_{B_R}|v-u|\mathrm{d}x,
\end{aligned} \tag{37}
$$

where $\xi\in(\min\{u^+,v^+\},\max\{u^+,v^+\})\subset(0,\|u\|_{L^\infty(\Omega)})$ and C is independent of R.

Similarly, we get by [20, (16) on page 44]

$$
\int_{B_R}|v-u|\mathrm{d}x \leq C(\varepsilon_1)R^{n+\alpha_1}+\varepsilon_1 R^{\beta_1}\int_{B_R}G(|\nabla v-\nabla u|)\mathrm{d}x, \tag{38}
$$

where $\alpha_1,\beta_1>0$ are independent of R, ε_1 will be chosen later.

By Lemma 12, (34), (35), (36), (37) and (38), we always have

$$
\begin{aligned}
\int_{B_R}G(|\nabla u-\nabla v|)\mathrm{d}x \leq& CR^{n+\alpha_2}+C\varepsilon_2R^{\beta_2}\int_{B_R}G(|\nabla u-\nabla v|)\mathrm{d}x + CR^{\frac{\lambda}{2}+\frac{n+\alpha_2}{2}} \\
&+C\varepsilon_2^{\frac{1}{2}}R^{\frac{\lambda}{2}+\frac{\beta_2}{2}}\left(\int_{B_R}G(|\nabla u-\nabla v|)\mathrm{d}x\right)^{\frac{1}{2}} \\
\leq& CR^{n+\alpha_2}+C\varepsilon_2R^{\beta_2}\int_{B_R}G(|\nabla u-\nabla v|)\mathrm{d}x + CR^{\frac{\lambda+n+\alpha_2}{2}}+CR^{\lambda+\beta_2}
\end{aligned}
$$

$$+ \varepsilon_2 \int_{B_R} G(|\nabla u - \nabla v|) \mathrm{d}x,$$

where $\alpha_2 = \min\{\alpha_0, \alpha_1\}, \beta_2 = \min\{\beta_0, \beta_1\}, \varepsilon_2 = \max\{\varepsilon_0, \varepsilon_1\}$.

Choosing ε_2 small enough, we get

$$\int_{B_R} G(|\nabla u - \nabla v|) \mathrm{d}x \leq C R^m, \tag{39}$$

where $m = \min\{n + \alpha_2, \frac{\lambda + n + \alpha_2}{2}, \lambda + \beta_2\}$.

Finally, we get by (34), (35), (36), (37), (38) and (39)

$$\int_{B_R} (G(|\nabla u|) - G(|\nabla v|)) \mathrm{d}x \leq C R^{n+\alpha_2} + C R^{\beta_2 + m}. \tag{40}$$

Putting (40) into (33), we obtain for all $0 < r \leq R$

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|) \mathrm{d}x \leq C \left(\frac{r}{R}\right)^{n+\sigma} \int_{B_R} G(|\nabla u - (\nabla u)_R|) \mathrm{d}x + C R^{n+\alpha_2} + C R^{\beta_2 + m} + C R^{\frac{\lambda}{2} + \frac{n+\alpha_2}{2}} + C R^{\frac{\lambda}{2} + \frac{\beta_2 + m}{2}}.$$

Due to the arbitrariness of $\lambda \in (0, n)$, we get $\min\{\beta_2 + m, \frac{\lambda}{2} + \frac{n+\alpha_2}{2}, \frac{\lambda}{2} + \frac{\beta_2+m}{2}\} > n$ by setting $\min\{\lambda + \alpha_2, \lambda + \beta_2\} > n$. We conclude that there exists $\alpha_3 > 0$ such that

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|) \mathrm{d}x \leq C \left(\frac{r}{R}\right)^{n+\sigma} \int_{B_R} G(|\nabla u - (\nabla u)_R|) \mathrm{d}x + C R^{n+\alpha_3}.$$

In view of Lemma 14, we conclude that there is a constant $\alpha_4 \in (0, 1)$ such that

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|) \mathrm{d}x \leq C r^{n+\alpha_4}. \tag{41}$$

Proceeding exactly as in [20, (22) on page 46)], we conclude that there is a constant $\alpha \in (0, 1)$ such that

$$\int_{B_r} |\nabla u - (\nabla u)_r| \mathrm{d}x \leq C r^{n+\alpha}, \tag{42}$$

which and Campanato's Embedding Theorem give the Hölder continuity of the gradient of u. ∎

Proof of Theorem 19 For any fixed $x_0 \in \Omega$, let $R > 0$ such that $R < \mathrm{dist}(x_0, \partial\Omega)$. As before, we denote $B_R = B_R(x_0)$. Let h be the $G-$harmonic function in B_R that agrees with u on the boundary, i.e.,

$$\mathrm{div} \, \frac{g(|\nabla h|)}{|\nabla h|} \nabla h = 0 \text{ in } B_R \text{ and } h - u \in W_0^{1,G}(B_R).$$

It suffices to note that $\int_{B_R} (\lambda_+ \chi_{\{h>0\}} - \lambda_+ \chi_{\{u>0\}}) \mathrm{d}x \leq \lambda_+ R^n$. Proceeding as in the proof of Theorem 18, we have

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|) \mathrm{d}x \leq C \left(\frac{r}{R}\right)^{n+\sigma} \int_{B_R} G(|\nabla u - (\nabla u)_R|) \mathrm{d}x + C R^n,$$

for all $0 < r \leq R$. Then Lemma 14 gives

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|) \mathrm{d}x \leq C r^n.$$

Finally,

$$\int_{B_r} |\nabla u - (\nabla u)_r| \mathrm{d}x \leq C r^n,$$

which shows that the gradient of u lies in BMO space and for any fixed subdomain $\Omega' \Subset \Omega$, there holds $\|u\|_{BMO(\Omega')} \leq C$ for a universal constant $C > 0$. The residual argument is the same as in [14, Section 5], and the desired result can be obtained. ■

5 Growth rates near the free boundary for nonnegative minimizers of $\mathcal{J}(u)$

In view of Corollary 17 or Remark 4, we may consider non-negative minimizers of $\mathcal{J}(u)$ and establish their growth rates near the free boundary $\partial\{u > 0\}$ for $\lambda_+ = 0$ and $\lambda_+ > 0$ respectively.

Theorem 20 (Growth rates for $\lambda_+ = 0$) *Let u be a non-negative minimizer of $\mathcal{J}(u)$ with $\lambda_+ = 0$. Assume taht $x_0 \in \partial\{u > 0\}$ and $B_{r_0}(x_0) \Subset \Omega$. Then there exists universal constants C_0, C_1, depending only on $n, \theta_0, f_0, \delta_0, g_0, G(1), \frac{1}{G(1)}, \|u\|_{L^\infty(\Omega)}$, $\|h\|_{L^\infty(\Omega)}, \|q\|_{L^\infty(\Omega)}, \|\varphi\|_{L^\infty(\Omega)}$ and $B_{r_0}(x_0)$, such that*

$$|u(x)| \leq C_0 \Phi(|x - x_0|), \quad \forall\, x \in B_{r_0}(x_0), \tag{43}$$
$$|\nabla u(x)| \leq C_1 \Phi'(|x - x_0|), \quad \forall\, x \in B_{r_0}(x_0). \tag{44}$$

for all $0 < r < r_0$, where $\Phi(t) = t^{p_0}$ with $p_0 = \min\{\frac{1+g_0}{g_0-\theta_0}, \frac{1+\delta_0}{\delta_0-\theta_0}, \frac{1+g_0}{g_0}\} > 1$.

Theorem 21 (Growth rates for $\lambda_+ > 0$) *Let u be a non-negative minimizer of $\mathcal{J}(u)$ with $\lambda_+ > 0$. Assume that $x_0 \in \partial\{u > 0\}$ and $B_{r_0}(x_0) \Subset \Omega$. Then there exists a universal constant C_2, depending only on $n, \theta_0, f_0, \delta_0, g_0, G(1), \frac{1}{G(1)}, \|u\|_{L^\infty(\Omega)}$, $\|h\|_{L^\infty(\Omega)}, \|q\|_{L^\infty(\Omega)}, \|\varphi\|_{L^\infty(\Omega)}$, and $_{r_0}(x_0)$ such that*

$$|u(x)| \leq C_2 |x - x_0|, \quad \forall\, x \in B_{r_0}(x_0), \tag{45}$$

for all $0 < r < r_0$.

Proof of Theorem 20 Due to the local property, we may assume that u is a non-negative minimizer of $\mathcal{J}(u)$ associated with the domain $B_1(x_0)$ with $x_0 = 0$. Firstly, we prove (43). Let $S(j,u) = \sup\limits_{x \in B_{2^{-j}}} |u(x)|$. It suffices to show that for all $j \in \mathbb{N}$ there holds

$$S(j+1, u) \leq \max\left\{c\Phi(2^{-j}), S(j,u)\Phi(2^{-1}), ..., S(j-m,u)\Phi(2^{-(m+1)}), ..., S(0,u)\Phi(2^{-j-1})\right\}, \tag{46}$$

with some constant $c > 0$. We prove by contradiction. Let us suppose (46) fails. Then for any $k \in \mathbb{N}$, there exists a sequence of integers $j_k \in \mathbb{N}$ and a sequence of minimizers u_k such that

$$S(j_k+1, u_k) > \max\left\{k\Phi(2^{-j_k}), S(j_k,u_k)\Phi(2^{-1}), ..., S(j_k-m,u_k)\Phi(2^{-(m+1)}), ..., S(0,u_k)\Phi(2^{-j_k-1})\right\}. \tag{47}$$

Notice that by (47) and the boundedness of u_k, it follows that $j_k \to \infty$ as $k \to \infty$.

Let $v_k(x) = \frac{u_k(2^{-j_k}x)}{S(j_k+1,u_k)}$, $\sigma_k = 2^{j_k} S(j_k+1, u_k)$, $G_k(t) = \frac{G(\sigma_k t)}{\sigma_k g(\sigma_k)}$ with $g_k(t) = G_k'(t)$, $F_k(t) = \frac{F(S(j_k+1,u_k)t)}{\sigma_j g(\sigma_k)}$ with $f_k(t) = F_k'(t)$, $q_k(x) = q(2^{-j_k}x)$ and $h_k(x) = \frac{S(j_k+1,u_k)}{\sigma_k g(\sigma_k)} h(2^{-j_k}x)$. By (G_5), G_k and F_k satisfy (1) and (2) with the same constants δ_0, g_0, θ_0 andf_0. For all $k > 0$, v_k is a minimizer of the functional $\int_{B_{2^{j_k}}} \big(G_k(|\nabla v|) + q_k F_k(v^+) + h_k v\big) \mathrm{d}x$. Indeed, by a simple calculation we have

$$\int_{B_{2^{j_k}}} \big(G_k(|\nabla v_k|) + q_k F_k(v_j^+) + h_k v_k\big) \mathrm{d}x = \frac{2^{j_k n}}{\sigma_k g(\sigma_k)} \int_{B_1} \big(G(|\nabla u|) + qF(u^+) + hu\big) \mathrm{d}x.$$

Particularly, v_k is a minimizer of the following functional

$$\mathcal{J}_k = \int_{B_R} \big(G_k(|\nabla v|) + q_k F_k(v^+) + h_k v\big)\mathrm{d}x,$$

provided $R = 2^m < 2^{j_k}$, m is fixed.

Notice that by (47) and the definition of Φ, we have $\sup\limits_{B_R} |v_k| \leq C\Phi(R)$, and

$$S(j_k+1, u_k) \geq k\Phi(2^{-j_k}) = k(2^{-j_k})^{p_0},$$

which gives

$$\begin{aligned} 2^{(1+\delta_0)j_k}(S(j_k+1,u_k))^{\delta_0-\theta_0} \geq & k^{\delta_0-\theta_0}2^{(1+\delta_0)j_k}(2^{-j_k})^{p_0(\delta_0-\theta_0)} \\ = & (2^{j_k})^{1+\delta_0-p_0(\delta_0-\theta_0)}k^{\delta_0-\theta_0} \\ \geq & k^{\delta_0-\theta_0}, \end{aligned} \tag{48}$$

and

$$\begin{aligned} 2^{(1+\delta_0)j_k}(S(j_k+1,u_k))^{g_0-\theta_0} \geq & k^{g_0-\theta_0}2^{(1+g_0)j_k}(2^{-j_k})^{p_0(g_0-\theta_0)} \\ \geq & k^{g_0-\theta_0}, \end{aligned} \tag{49}$$

Then we get by (F_1), (G_2), (G_3), (48), (49) and $\sup\limits_{B_R} |v_k| \leq C\Phi(R)$

$$\begin{aligned} |q_k F_k(v_k^+)| = & |q_k| \frac{F(S(j_k+1,u_k)v_k^+)}{\sigma_k g(\sigma_k)} \\ \leq & \frac{C|S(j_k+1,u_k)|^{1+\theta_0}F(v_k^+)}{\sigma_k g(\sigma_k)} \\ \leq & \frac{C|S(j_k+1,u_k)|^{1+\theta_0}}{\min\{(2^{j_k}S(j_k+1,u_k))^{1+\delta_0}, (2^{j_k}S(j_k+1,u_j))^{1+g_0}\}} \\ \leq & \frac{C}{\min\{2^{(1+\delta_0)j_k}(S(j_k+1,u_k))^{\delta_0-\theta_0}, (2^{(1+g_0)j_k}(S(j_k+1,u_k))^{g_0-\theta_0}\}} \\ \leq & \frac{C}{\min\{k^{\delta_0-\theta_0}, k^{g_0-\theta_0}\}} \to 0, \quad \text{as } k\to\infty. \end{aligned} \tag{50}$$

Similarly, due to that $2^{(1+\delta_0)j_k}(S(j_k+1,u_k))^{\delta_0} \geq k^{\delta_0}$, and $2^{(1+\delta_0)j_k}(S(j_k+1,u_k))^{g_0} \geq k^{g_0}$, we have $|h_k| \to 0$ as $k \to \infty$. For k large enough, according to the $C^{1,\alpha}$ regularity of minimizers, we obtain $\|v_k\|_{C^{1,\alpha}(B_R)} \leq C$ (see Theorem 18). Note that C depends on $\frac{1}{G_k(1)}$ and $G_k(1)$. However by (G_5), we see that C depends on $\frac{1}{G(1)}$ and $G(1)$ essentially, thus it is independent of k. Therefore, up to subsequence, we get $v_k \to v_0$ in $C^{1,\beta}(B_{r_0})$ with $0<\beta<\alpha$ and any $r_0 < 1$. We deduce by $v_k(0)=0$ and $\sup\limits_{B_{\frac{1}{2}}} |v_k| = 1$ that

$$\sup_{B_{\frac{1}{2}}} |v_0| = 1, v_0(0) = 0, \tag{51}$$

On the other hand, using the compact condition (5), we conclude that (see [5, Theorem 6.1]) there exists a function $G_\infty \in C^2(0,+\infty)$ such that, up to a subsequence,

$$\begin{aligned} & G_k \to G_\infty, \ \ g_k = G_k' \to G_\infty' = g_\infty \ \text{ uniformly in compact subsets of } [0,+\infty), \\ & G_k'' \to G_\infty'' \ \text{ uniformly in compact subsets of } (0,+\infty), \end{aligned}$$

and g_∞ satisfies structural condition (1) with the same constants. Furthermore, we infer that v_0 is a $G_\infty-$harmonic function in B_1. Since $v_k \geq 0$ in B_1, $v_0 \geq 0$ in B_{r_0}. Recalling $v_0(0) = 0$ and the Harnack's inequality, we have $v_0 \equiv 0$ in B_{r_0}. Finally we get $v_0 \equiv 0$ in B_1 due to the continuity of v_0 and the arbitrariness of r_0. which is a contradiction with (51). Therefore have proved (43).

Now, we prove (44). Set $S(j, |\nabla u|) = \sup\limits_{x\in B_{2^{-j}}} |\nabla u(x)|$. It suffices to show

$$S(j+1, |\nabla u|) \leq \max\Big\{c\Phi'(2^{-j}), S(j, |\nabla u|)\Phi'(2^{-1}),..., S(j-m, |\nabla u|)\Phi'(2^{-(m+1)}), \\ ..., S(0, |\nabla u|)\Phi'(2^{-j-1})\Big\}. \tag{52}$$

for some positive constant c. By contradiction, suppose that (52) fails. Then for any $k \in \mathbb{N}$, there exists a sequence of integers j_k and a sequence of minimizers u_k such that

$$S(j_k+1, |\nabla u_k|) > \max\big\{k\Phi'(2^{-j_k}), S(j_k, |\nabla u_k|)\Phi'(2^{-1}),..., S(j_k-m, |\nabla u_k|)\Phi'(2^{-(m+1)}), \\ ..., S(0, |\nabla u_k|)\Phi'(2^{-j_k-1})\big\}. \tag{53}$$

Let $v_k(x) = \frac{u_k(2^{-j_k}x)}{2^{-j_k}S(j_k+1,u_k)}$, $\varrho_k = S(j_k+1, u_k)$, $G_k(t) = \frac{G(\varrho_k t)}{\varrho_k g(\varrho_k)}$ with $g_k(t) = G_k'(t)$, $F_k(t) = \frac{F(S(j_k+1,u_k)t)}{\sigma_j g(\varrho_k)}$ with $f_k(t) = F_k'(t)$, $q_k(x) = q(2^{-j_k}x)$ and $h_k(x) = \frac{S(j_k+1,u_k)}{\varrho_k g(\varrho_k)} h(2^{-j_k}x)$. Then for all $k > 0$, v_k is a minimizer of the functional $\int_{B_{2^{j_k}}} \big(G_k(|\nabla v|) + q_k F_k(v^+) + h_k v\big)\mathrm{d}x$. By (43) and (53), we have $\sup\limits_{B_1} |v_k| \leq \frac{C}{k} \to 0$ as $k \to \infty$. Arguing as before, we get $|q_k F_k(v_k^+)| \to 0$ and $|h_k| \to 0$ as $k \to \infty$, and we can conclude that there exists a $G_\infty-$harmonic function v_0 in B_1, satisfying $v_k \to v_0$ in $C^{1,\beta}(B_{r_0})$ with some $\beta \in (0, 1)$ and any $r_0 < 1$. Furthermore, we conclude that $v_0 \equiv 0$ in B_1. However, note that $\sup\limits_{B_{1/2}} |\nabla v_{j_k}| = 1$. Thus $\sup\limits_{B_{1/2}} |\nabla v_0| = 1$, which gives a contradiction. ■

Proof of Theorem 21 Let $\Phi(t) = t^{p_0}$ for all $t \geq 0$, where $p_0 = \min\{\frac{1+g_0}{g_0-\theta_0}, \frac{1+\delta_0}{\delta_0-\theta_0}, \frac{1+g_0}{g_0}, 1\} = 1$. Then one can proceed with a slight modification of the proof of Theorem 20 to obtain $|u(x)| \leq C_2\Phi(|x - x_0|)$. ■

Corollary 22 (Optimal growths in the non-homogenous one-phase problems for $p-$Laplacian) *Let $G(t) = t^p$ with $p > 1$, and $F(t) = t^\gamma$ with $0 < \gamma < p$. Let u be a nonnegative minimizer of $\mathcal{J}(u)$ with $\lambda+ = 0$ in* (3). *Assume that $x_0 \in \partial\{u > 0\}$. Then there exists a universal constant C such that*

$$|u(x)| \leq C|x - x_0|^{p_0}, \; |\nabla u(x)| \leq C|x - x_0|^{p_0-1}, \quad \forall\, x \in B_{r_0}(x_0) \Subset \Omega$$

for all $0 < r < r_0$, where $p_0 = \min\{\frac{p}{p-\gamma}, \frac{p}{p-1}\} > 1$.

Remark 5 *Checking the proof of Theorem 20, if $h = 0$ and u is a nonnegative minimizer of $\mathcal{J}(u)$ with $\lambda+ = 0$ in* (3), *then we have*

$$|u(x)| \leq C|x - x_0|^{p_1}, \; |\nabla u(x)| \leq C|x - x_0|^{p_1-1}, \quad \forall\, x \in B_{r_0}(x_0) \Subset \Omega$$

where $p_1 = \min\{\frac{1+g_0}{g_0-\theta_0}, \frac{1+\delta_0}{\delta_0-\theta_0}\}$. Particularly, if $G(t) = t^p, p > 1$ and $F(t) = t^\gamma, 0 < \gamma < p$, we have

$$|u(x)| \leq C|x - x_0|^{\frac{p}{p-\gamma}}, \; |\nabla u(x)| \leq C|x - x_0|^{\frac{\gamma}{p-\gamma}}, \quad \forall\, x \in B_{r_0}(x_0) \Subset \Omega,$$

which are the optimal growth rates of minimizers and their gradients in the homogeneous one-phase free boundary problems for $p-$Laplacian.

Remark 6 *Condition* (5) *is used only for the compactness of G_k by blow-up techniques, see, e.g., [5, Theorem 6.1]. For the case of $p-$Laplacian, i.e., $G(t) = t^p$,* (5) *becomes trivial due to that $Q'(s) \equiv 0$.*

6 Local Lipschitz continuity of non-negative minimizers of $\mathcal{J}(u)$ with $\lambda_+ > 0$

In this section, in order to obtain local Lipschitz continuity of non-negative minimizers of $\mathcal{J}(u)$ with $\lambda_+ > 0$, we make further assumptions on F, i.e., assume that $F \in C^1([0,+\infty);[0,+\infty))$. Note that $f \in C([0,+\infty);[0,+\infty))$ and there exists positive constants C_1 and C_2 such that $f(t) \le C_1 + C_2 g(t)$ for all $t \ge 0$.

We say that a function $u \in W^{1,G}(D)$ is a weak solution of the equation div $\frac{g(|\nabla u|)}{|\nabla u|}\nabla u = qf(u)$ in D, if

$$\int_D \frac{g(|\nabla u|)}{|\nabla u|}\nabla u \nabla \xi \mathrm{d}x + \int_D (qf(u)+h)\xi \mathrm{d}x = 0 \tag{54}$$

holds for all $\xi \in W_0^{1,\tilde{G}}(D)$, where D is a domain.

Lemma 23 (Harnack's inequality) *Let $u \in W^{1,G}(B_R)$ with $0 \le u \le M$ is a weak solution of div $\frac{g(|\nabla u|)}{|\nabla u|}\nabla u = qf(u) + h$ in B_R. Then, there exists a universal constant $t_0 > 0$ and a constant $C_r > 0$ depending only on $\delta_0, g_0, M, t_0, \|q\|_{L^\infty(B_R)}$ and $R - r$ such that*

$$\sup_{B_r} u \le C_r\left(\inf_{B_r} u + g^{-1}(R)R\right),$$

for all $0 < r \le R$.

Proof It is a direct result of [16, Corollary 1.4]. Indeed, we amy set $a_1 = a_2 = a_4 = a_5 = 0$ and $a_3 = 1$ in (1.3a) and (1.3b) of [16, Corollary 1.4] for our problem. We shall verify that conditions (1.3c)" and (1.4) of [16, Corollary 1.4] are satisfied. By (f_2), there exits $t_0 > 0$ such that for all $t > t_0$, there holds

$$\begin{aligned} f(t) \le & g(t) = g\left(\frac{t}{R}\cdot R\right) \\ \le & g\left(\frac{t}{R}\right)\frac{t}{R}\frac{R}{t}\max\{R^{\delta_0}, R^{g_0}\} \quad \text{by } (g_1) \\ \le & \frac{1}{t_0}\cdot g\left(\frac{t}{R}\right)\frac{t}{R}, \end{aligned}$$

where without loss of generality we assume that $R \le 1$. For $0 \le t \le t_0$, due to $F \in C^1([0,+\infty);[0,+\infty))$, $f(t) = F'(t)$ is continuous in $[0, t_0]$. Then there exists a constant $M_0 > 0$ such that $f(t) \le M_0$ for all $t \in [0, t_0]$. Then for all $t \ge 0$, we have

$$f(t) \le \frac{1}{t_0}\cdot g\left(\frac{t}{R}\right)\frac{t}{R} + M_0.$$

Thus we can choose $b_0 = 0, b_1 = \frac{1}{t_0}\cdot\|q\|_{L^\infty(B_R)}, b_2 = M_0\cdot\|q\|_{L^\infty(B_R)} + \|h\|_{L^\infty(B_R)}$ and $\chi = g^{-1}(b_2 R)$ in (1.3c)" and (1.4) of [16, Corollary 1.4]. Finally, by $(\tilde{g}_1)$ and [16, Corollary 1.4], we have

$$\sup_{B_r} u \le C\left(\inf_{B_r} u + g^{-1}(b_2R)R\right) \le C'\left(\inf_{B_r} u + g^{-1}(R)R\right).$$

■

A consequence of Theorem 16 is the fact that $\{u > 0\}$ is an open set. We have the following result.

Lemma 24 *Let u be a non-negative minimizer of $\mathcal{J}(u)$ with $\lambda_+ > 0$ in (3). Then u is a weak solution of the following equation*

$$div\ \frac{g(|\nabla u|)}{|\nabla u|}\nabla u = qf(u) + h \quad in\ \{u > 0\}.$$

Proof For any ball $B \subset \{u > 0\}$, consider first that $\xi \in C_0^\infty(B)$. There exists $0 < \varepsilon_0 \leq 1$ small enough such that $\{u \pm \varepsilon\xi > 0\} \cap B = B$ for all $0 < \varepsilon \leq \varepsilon_0$. Standard arguments implies that

$$\lim_{\varepsilon \to 0+} \int_B \frac{F((u+\varepsilon\xi)^+) - F(u^+)}{\varepsilon} \mathrm{d}x = \int_B f(u)\xi \mathrm{d}x. \tag{55}$$

The minimality of u implies that

$$\begin{aligned} 0 \leq & \frac{1}{\varepsilon} \int_B \Big(G(|\nabla(u+\varepsilon\xi)|) - G(|\nabla u|) + q(F((u+\varepsilon\xi)^+) - F(u^+)) \Big) \mathrm{d}x \\ \leq & \int_B g(|\nabla u + \varepsilon\nabla\xi|) \frac{\nabla u + \varepsilon\nabla\xi}{|\nabla u + \varepsilon\nabla\xi|} \nabla\xi \mathrm{d}x + \frac{1}{\varepsilon} \left(\int_B q(F((u+\varepsilon\xi)^+) - F(u^+)) \mathrm{d}x + \int_B h\varepsilon\xi \mathrm{d}x \right), \end{aligned} \tag{56}$$

where in the last inequality we used the convexity of G.

From (55), (56) and letting $\varepsilon \to 0^+$, we get

$$\int_B \frac{g(|\nabla u|)}{|\nabla u|} \nabla u \nabla\xi \mathrm{d}x + \int_B (qf(u) + h)\xi \mathrm{d}x \geq 0. \tag{57}$$

Using the function $\phi = u - \varepsilon\xi$ and repeating the previous arguments we get

$$-\int_B \frac{g(|\nabla u|)}{|\nabla u|} \nabla u \nabla\xi \mathrm{d}x - \int_B (qf(u) + h)\xi \mathrm{d}x \geq 0, \tag{58}$$

for all $\xi \in C_0^\infty(B)$. By (57) and (58), (54) holds for all $\xi \in C_0^\infty(B)$. Now for $\xi \in W_0^{1,\tilde{G}}(B)$, let $\xi_n \in C_0^\infty(B)$ with $\xi_n \to \xi$ in $W_0^{1,\tilde{G}}(B)$ as $n \to \infty$, then (54) holds with $\xi_n \in C_0^\infty(B)$. We conclude the desired result by letting $n \to \infty$ and the arbitrariness of B. ■

Theorem 25 (Local Lipschitz continuity for $\lambda_+ > 0$) *Given a subdomain $\Omega' \Subset \Omega$, there exists a constant $C > 0$ that depends only on Ω' and universal constants, such that for any nonnegative minimizer of $\mathcal{J}(u)$ with $\lambda_+ > 0$ in* (3)*, there holds*

$$\|\nabla u\|_{L^\infty(\Omega')} \leq C. \tag{59}$$

Proof We proceed as the proof of [15, Theorem 4.1], supposing that (59) fails. Then there exists a sequence of points $X_j \in \Omega' \cap \{u > 0\}$ such that

$$X_j \to \partial\{u > 0\} \quad \text{and} \quad \frac{u(X_j)}{\mathrm{dist}(X_j, \partial\{u > 0\})} \to +\infty \quad \text{as } j \to +\infty. \tag{60}$$

Denote $U_j = u(X_j)$ and $d_j = \mathrm{dist}(X_j, \partial\{u > 0\})$. Let $Y_j \in \partial\{u > 0\}$ satisfying $d_j = |X_j - Y_j|$. Note that we have

$$\mathrm{div}\, \frac{g(|\nabla u|)}{|\nabla u|} \nabla u = qf(u) + h \quad \text{in } \{u > 0\}.$$

Thus, by Harnack's inequality, $(\tilde{g}_1)$, and the boundedness of u, there exists a constant c depending only on Ω' and universal constants, such that

$$d_j + \inf_{\overline{B}_{\frac{3}{4}d_j}(X_j)} u \geq cU_j.$$

In turn, we have

$$\sup_{\overline{B}_{\frac{d_j}{4}}(Y_j)} u \geq cU_j - d_j. \tag{61}$$

Consider the set $A_j = \left\{Z \in B_{d_j}(Y_j) : \text{dist}(Z, \partial\{u > 0\}) \leq \frac{1}{3}\text{dist}(Z, \partial B_{d_j}(Y_j))\right\}$. Then $B_{\frac{d_j}{4}}(Y_j) \subset A_j$ (see the proof of [15, Theorem 4.1]). Thus

$$\begin{aligned}
\text{dist}(Z_j, \partial B_{d_j}(Y_j))u(Z_j) :=& M_j \\
:=& \sup_{Z \in A_j} \text{dist}(Z, \partial B_{d_j}(Y_j))u(Z) \\
\geq& \sup_{Z \in \overline{B}_{\frac{d_j}{4}}(Y_j)} \text{dist}(Z, \partial B_{d_j}(Y_j))u(Z) \\
\geq& \sup_{Z \in \overline{B}_{\frac{d_j}{4}}(Y_j)} \frac{3d_j}{4} u(Z) \\
=& \frac{3d_j}{4} \sup_{\overline{B}_{\frac{d_j}{4}}(Y_j)} u.
\end{aligned}$$

It follows that

$$u(Z_j) \geq \frac{d_j}{\text{dist}(Z_j, \partial B_{d_j}(Y_j))} \frac{3}{4} \sup_{\overline{B}_{\frac{d_j}{4}}(Y_j)} u \geq \frac{3}{4} \sup_{\overline{B}_{\frac{d_j}{4}}(Y_j)} u.$$

Using (61), we have

$$u(Z_j) \geq \frac{3}{4}(cU_j - d_j). \tag{62}$$

For each j, let $W_j \in \partial\{u > 0\}$ satisfy

$$r_j = |Z_j - W_j| = \text{dist}(Z_j, \partial\{u > 0\}) \leq \frac{1}{3}\text{dist}(Z_j, \partial B_{d_j}(Y_j)).$$

One may get (see (4.7) in [15])

$$\frac{d_j}{r_j} \geq 4. \tag{63}$$

It follows from (62), (63) and (60) that

$$\frac{u(Z_j)}{r_j} \geq \frac{3d_j}{4r_j}\left(c\frac{U_j}{d_j} - 1\right) \geq 3\left(c\frac{U_j}{d_j} - 1\right) \to +\infty. \tag{64}$$

Proceeding as (4.10), (4.11) in [15], one has (for j large enough)

$$\sup_{B_{\frac{r_j}{2}}(W_j)} u \leq 2u(Z_j), \qquad \sup_{\overline{B}_{\frac{r_j}{4}}(W_j)} \frac{u}{u(Z_j)} \geq \frac{c'}{2}, \tag{65}$$

for some universal constant $c' > 0$. Now for each j, define the function $u_j : B_1(0) \to (0, 2)$ by

$$u_j(X) = \frac{u(W_j + \frac{r_j}{2}X)}{u(Z_j)}. \tag{66}$$

It follows from (65) that

$$\max_{B_1(0)} u_j \leq 2, \quad \max_{B_1(0)} u_j \geq \frac{c'}{2}, \quad u_j(0) = 0. \tag{67}$$

Let $\sigma_j=\frac{2u(Z_j)}{r_j}$, $G_j(t)=\frac{G(\sigma_j t)}{\sigma_j g(\sigma_j)}$ with $g_j(t)=G_j'(t)$, $F_j(t)=\frac{F(u(Z_j)t)}{\sigma_j g(\sigma_j)}$ with $f_j(t)=F_j'(t)$, $q_j(X)=q(W_j+\frac{r_j}{2}X)$, $h_j(X)=\frac{h(W_j+\frac{r_j}{2}X)}{\sigma_j g(\sigma_j)}$ and $\lambda_{+j}=\frac{\lambda_+}{\sigma_j g(\sigma_j)}$. Then for all $j>0$,

$$\delta_0\leq\frac{tg_j'(t)}{g_j(t)}\leq g_0,\qquad 1+\theta_0\leq\frac{tF_j'(t)}{F_j(t)}\leq 1+f_0. \tag{68}$$

Let v be the $G-$harmonic function in $B_{\frac{r_j}{2}}(W_j)$ with the boundary data u, i.e.,

$$\begin{cases}\text{div }\frac{g(|\nabla v|)}{|\nabla v|}\nabla v=0 & \text{in } B_{\frac{r_j}{2}}(W_j),\\ \qquad\qquad v=u & \text{on } \partial B_{\frac{r_j}{2}}(W_j).\end{cases}$$

Let $v_j:B_1(0)\to(0,2)$ be defined by $v_j(X)=\frac{v(W_j+\frac{r_j}{2}X)}{u(Z_j)}$. Then v_j satisfies

$$\begin{cases}\text{div }\frac{g_j(|\nabla v_j|)}{|\nabla v_j|}\nabla v_j=0 & \text{in } B_1(0),\\ \qquad\qquad v_j=u_j & \text{on } \partial B_1(0).\end{cases}$$

Let $Y=W_j+\frac{r_j}{2}X$, then

$$\int_{B_{\frac{r_j}{2}}(W_j)}G(|\nabla u(Y)|)\mathrm{d}Y=\left(\frac{r_j}{2}\right)^n\int_{B_1(0)}G(\sigma_j|\nabla u_j(X)|)\mathrm{d}X.$$

It follows

$$\begin{aligned}\int_{B_{\frac{r_j}{2}}(W_j)}\frac{G(|\nabla u(Y)|)}{\sigma_j g(\sigma_j)}\mathrm{d}Y=&\left(\frac{r_j}{2}\right)^n\int_{B_1(0)}\frac{G(\sigma_j|\nabla u_j(X)|)}{\sigma_j g(\sigma_j)}\mathrm{d}X\\ =&\left(\frac{r_j}{2}\right)^n\int_{B_1(0)}G_j(|\nabla u_j(X)|)\mathrm{d}X,\end{aligned}$$

which gives

$$\int_{B_1(0)}G_j(|\nabla u_j|)\mathrm{d}x=\left(\frac{r_j}{2}\right)^{-n}\int_{B_{\frac{r_j}{2}}(W_j)}\frac{G(|\nabla u|)}{\sigma_j g(\sigma_j)}\mathrm{d}x. \tag{69}$$

By the minimality of u, we have

$$\begin{aligned}\int_{B_{\frac{r_j}{2}}(W_j)}G(|\nabla u|)\mathrm{d}x-\int_{B_{\frac{r_j}{2}}(W_j)}G(|\nabla v|)\mathrm{d}x\leq&\int_{B_{\frac{r_j}{2}}(W_j)}\Big(q(F(v^+)-F(u^+))+h(v-u)\Big)\mathrm{d}x\\ &+\lambda_+\int_{B_{\frac{r_j}{2}}(W_j)}(\chi_{\{h>0\}}-\chi_{\{u>0\}})\mathrm{d}x\\ \leq&\|q\|_{L^\infty(B_{\frac{r_j}{2}})}\int_{B_{\frac{r_j}{2}}(W_j)}(|F(h)|+|F(u)|)\mathrm{d}x+Cr_j^n\\ \leq&Cr_j^n,\end{aligned} \tag{70}$$

where we used the boundedness of h and u,v, and the increasing property of F in the last inequality.

We infer from (69) and (70)

$$\int_{B_1(0)} G_j(|\nabla u_j|)\mathrm{d}x - \int_{B_1(0)} G_j(|\nabla v_j|)\mathrm{d}x \le \frac{C}{\sigma_j g(\sigma_j)} \to 0 \ \text{ by } \sigma_j \to +\infty. \tag{71}$$

Then we deduce by Lemma 12 and (71)

$$\int_{B_1} G_j(|\nabla u_j - \nabla v_j|)\mathrm{d}x \le C\Big(\frac{1}{\sigma_j g(\sigma_j)} + \frac{1}{\sqrt{\sigma_j g(\sigma_j)}}\Big) \to 0 \ \text{ as } j \to +\infty, \tag{72}$$

where we used the uniform boundedness of $\int_{B_1} G_j(|\nabla v_j|)\mathrm{d}x$ due to the uniform boundedness of u_j and v_j (see, e.g., Lemma 11).

We get by (G_3)

$$\int_{B_1^-} |\nabla u_j - \nabla v_j|^{1+g_0}\mathrm{d}x + \int_{B_1^+} |\nabla u_j - \nabla v_j|^{1+\delta_0}\mathrm{d}x \le C\int_{B_1} G_j(|\nabla u_j - \nabla v_j|)\mathrm{d}x, \tag{73}$$

where $B_1^- = B_1 \cap \{|\nabla u_j - \nabla v_j| < 1\}$ and $B_1^+ = B_1 \cap \{|\nabla u_j - \nabla v_j| \ge 1\}$. Hölder's inequality gives

$$\int_{B_1^-} |\nabla u_j - \nabla v_j|^{1+\delta_0}\mathrm{d}x \le C\Big(\int_{B_1^-} |\nabla u_j - \nabla v_j|^{1+g_0}\mathrm{d}x\Big)^{\frac{1+\delta_0}{1+g_0}}. \tag{74}$$

So we obtain by (73) and (74)

$$\Big(\int_{B_1^-} |\nabla u_j - \nabla v_j|^{1+\delta_0}\mathrm{d}x\Big)^{\frac{1+g_0}{1+\delta_0}} + \int_{B_1^+} |\nabla u_j - \nabla v_j|^{1+\delta_0}\mathrm{d}x \le C\int_{B_1} G_j(|\nabla u_j - \nabla v_j|)\mathrm{d}x,$$

which and (72) imply that

$$\int_{B_1} |\nabla u_j - \nabla v_j|^{1+\delta_0}\mathrm{d}x \to 0 \ \text{ by } j \to +\infty.$$

It follows by Poincaré's inequality that

$$u_j - v_j \to 0 \ \text{ strongly in } W_0^{1,1+\delta_0}(B_1). \tag{75}$$

Note that the uniform boundedness of v_j guarantees that, for any $r_0 \in (0,1)$ there exists a universal constant $C > 0$ satisfying $\|v_j\|_{C^{1,\alpha}(B_{r_0})} \le C$ (see, e.g. [20, Theorem 1.2]). Therefore, we can find v_0 in B_{r_0} such that, up to a subsequence,

$$v_k \to v_0 \text{ and } \nabla v_k \to \nabla v_0, \ \text{ uniformly in } B_{r_0}.$$

On the other hand, noting that u_j is a minimizer of the following functional

$$\mathcal{J}_j = \int_{B_1} (G_j(|\nabla w|) + q_j F_j(w^+) + h_j w + \lambda_{+j}\chi_{\{w>0\}})\mathrm{d}x \to \min,$$

and recalling the structural conditions of $g_j(t)$, $F_j(t)$, and λ_{+j}, and the boundedness of q_j, h_j, we have the uniform Hölder's estimate of u_j, i.e., $\|u_j\|_{C^\beta(B_{r_0})} \le C$. So, we conclude that there exists a $u_0 \in C^\beta(B_{r_0})$ such that

$$u_k \to u_0 \ \text{ uniformly in } B_{r_0}.$$

We conclude this way that $u_0 = v_0$ in B_{r_0} by (75).

Now using the compact condition (1), we conclude that (see [5, Theorem 6.1]) there exists a function $G_\infty \in C^2(0,+\infty)$ such that, up to a subsequence,

$$\begin{aligned}&G_j \to G_\infty, \ \ g_j = G_j' \to G_\infty' = g_\infty \ \text{ uniformly in compact } \text{ subsets of } [0,+\infty),\\&G_j'' \to G_\infty'' \ \text{ uniformly in compact } \text{ subsets of } (0,+\infty),\end{aligned}$$

and g_∞ satisfies the same structural condition as (68) with the same constants.

We now claim that u_0 is a $G_\infty-$harmonic function in B_{r_0}. Indeed, by the minimality of u_j again, we have for any $\varphi \in C_0^\infty(B_{r_0})$

$$\begin{aligned}\int_{B_{r_0}} G_j(|\nabla u_j|)\mathrm{d}x \le &\int_{B_{r_0}} \Big(G_j(|\nabla u_j+\nabla\varphi|) + q_jF_j((u_j+\varphi)^+) - q_jF_j(u_j^+) + h_j((u_j+\varphi)^+ - u_j^+)\Big)\mathrm{d}x\\&+\int_{B_{r_0}} (\lambda_{+j}\chi_{\{u_j+\varphi>0\}} - \lambda_{+j}\chi_{\{u_j>0\}})\mathrm{d}x. \end{aligned}\tag{76}$$

Note that q_j is uniformly bounded, $\sigma_j \to +\infty$, $\|h_j\|_{B_1} \to 0$ and u, φ are bounded, then

$$\|h_j((u_j+\varphi)^+ - u_j^+)\|_{B_1} \to 0, \tag{77}$$

$$q_jF_j(u_j^+) = \frac{q_jF\big(|u(W_j+\frac{r_j}{2}X)|\big)}{\sigma_j g(\sigma_j)} \le \frac{\|q\|_{L^\infty(B_1)}F\big(\sup\limits_{B_1}|u|\big)}{\sigma_j g(\sigma_j)} \to 0, \tag{78}$$

and

$$\begin{aligned}q_jF_j((u_j+\varphi)^+) =&\frac{\|q\|_{L^\infty(B_1)}F\big(u(W_j+\frac{r_j}{2}X)+u(Z_j)\varphi\big)}{\sigma_j g(\sigma_j)}\\ \le&\frac{\|q\|_{L^\infty(B_1)}F\big((1+\sup\limits_{B_1}|\varphi|)\sup\limits_{B_1}|u|\big)}{\sigma_j g(\sigma_j)} \to 0.\end{aligned}\tag{79}$$

Note also that

$$G_j(|\nabla u_j|) \le C(G_j(|\nabla u_j - \nabla v_j|) + G_j(|\nabla v_j|)),$$

which, the C^1-convergence of v_j, and (72) imply that there exists $\xi \in L^1(B_{r_0})$ such that

$$G_j(|\nabla u_j|) \le \xi \quad \text{a.e. in } B_{B_{r_0}}.$$

Once $\nabla u_j \to \nabla u_0$ a.e. in B_{r_0}, Lebesgue's dominated convergence theorem implies

$$\int_{B_{r_0}} G_j(|\nabla u_j|)\mathrm{d}x \to \int_{B_{r_0}} G_\infty(|\nabla u_0|)\mathrm{d}x,$$

and

$$\int_{B_{r_0}} G_j(|\nabla u_j+\nabla\varphi|)\mathrm{d}x \to \int_{B_{r_0}} G_\infty(|\nabla u_0+\nabla\varphi|)\mathrm{d}x.$$

Then we obtain by (76), (77), (78), (79), and $\lambda_{+j} \to 0$

$$\int_{B_{r_0}} G_\infty(|\nabla u_0)\mathrm{d}x \le \int_{B_{r_0}} G_\infty(|\nabla u_0+\nabla\varphi|)\mathrm{d}x.$$

This implies that u_0 is a $G_\infty-$harmonic function in B_{r_0}.

Since $u_j \geq 0$ in B_1, $u_0 \geq 0$ in B_{r_0}. Note that $u_0(0) = 0$. The Harnack's inequality implies $u_0 \equiv 0$ in B_{r_0}. Finally we get $u_0 \equiv 0$ in B_1 due to the continuity of u and the arbitrariness of r_0, which is a contradiction to (67). ■

References

[1] Adams, R. A., Fournier, J. J. F.: *Sobolev Spaces*. Pure and Applied Mathematics. Amsterdam: Acad. Press (140) 2003.

[2] Alt, H. M., Caffarelli, L. A., Friedman, A.: A free boundary problem for quasilinear elliptic equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, (4)11(1)(1984), 1-44.

[3] Alt, H. M., Phillips, D.: A free boundary problem for semilinear elliptic equations. *J. Reine Angew. Math.*, 368(1986), 63-107.

[4] Braga, J. E. M.: On the Lipschitz regularity and asymptotic behaviour of the free boundary for classes of minima of inhomogeneous two-phase Alt-Caffarelli functionals in Orlicz spaces. *Ann. Mat. Pura Appl.*, https://doi.org/10.1007/s10231-018-0755-7.

[5] Braga, J. E. M., Moreira, D. R.: Uniform Lipschitz regularity for classes of minimizers in two phase free boundary problems in Orlicz spaces with small density on the negative phase. *Ann. I. H. Poincaré- AN.*, 31(4)(2014), 823–850.

[6] Challal, S., Lyaghfouri, A.: Porosity of free boundaries in A-obstacle problems. *Non. Anal.: TMA.*, 70(7)(2009), 2772-2778.

[7] Challal, S., Lyaghfouri, A., Rodrigues, J. F.: On the A-obstacle problem and the Hausdorff measure of its free boundary. *Ann. Mat. Pura Appl.*, 191(2012), 113-165.

[8] Gossez, J.: Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients. *Trans. Am. Math. Soc.*, 190(1974), 163-205.

[9] Danielli, D., Petrosyan, A.: A minimum problem with free boundary for a degenerate quasilinear operator. *Calc. Var. Partial Differ. Equ.*, 23(1)(2005), 97-124.

[10] Giaquinta, M., Giusti, E.: Differentiability of minima of non-differentiable functionals. *Invent. Math.*, 72(1983), 285-298.

[11] Karp, L., Kilpeläinen, T., Petrosyan, A., Shahgholian, H.: On the porosity of free boundaries in degenerate variational inequalities. *J. Differ. Equ.*, 164(2000), 110-117.

[12] Ladyzhenskaya, O. A., Ural'tseva, N. N.: *Linear and Quasilinear Elliptic Equations*. vol. 46. Academic Press, New York, 1968.

[13] Lee, K., Shahgholian, H.: Hausdorff measure and stability for the p-obstacle problem $(2 < p < 1)$. *J. Differ. Equ.*, 195(2003), 14 – 24.

[14] Leitão, R., de Queiroz, O. S., Teixeira, E. V.: Regularity for degenerate two-phase free boundary problems. *Ann. Inst. H. Poincaré Anal. Non Lineare.*, 32(4)(2015), 741 – 762.

[15] Leitão, R., Teixeira, E. V.: Regularity and geometric estimates for minima of discontinuous functionals. *Rev. Mat. Iberoam.*, 31(1)(2015), 69 – 108.

[16] Lieberman, G. M.: The natural generalization of the natural conditions of Ladyzhensaya and Uraltseva for elliptic equations. *Comm. Partial Differ. Equ.*, 16 (2&3) (1991), 311 – 361.

[17] Martínez, S., Wolanski, N.: A minimum problem with free boundary in Orlicz spaces. *Adv. Math.*, 218(6) (2008), 1914 – 1971.

[18] Phillips, D.: A minimization problem and the regularity of solutions in the presence of a free boundary. *Indiana Univ. Math. J.*, 32(1983), 1 – 17.

[19] Phillips, D.: Hausdorff measure estimates of a free boundary for a minimum problem. *Comm. Partial Differ. Equ.*, 8(1983), 1409 – 1454.

[20] Zheng, J., Feng, B., Zhao, P.: Regularity of minimizers in the two-phase free boundary problems in Orlicz-Sobolev spaces. *Z. Anal. Anwend.*, 36(1)(2017), 37 – 47.

[21] Zheng, J., Guo, X.: Lyapunov-type inequalities for ψ - Laplacian equations. chinaXiv:201805. 00171.

[22] Zheng, J., Zhang, Z., Zhao, P.: A minimum problem with two-phase free boundary in Orlicz spaces. *Monatsh. Math.*, 172(3-4) (2013), 441 – 475.